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AN OPTIMIZATION MODEL FOR EXPLORING THE EGYPTIAN ROYAL PYRAMIDS LOCATIONS

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ABSTRACT

The specification of the factor used by the ancient Egyptians to locate their royal pyramids has been an age-old great interest of many archaeologists, some of them concluded that the reason behind the locating of the ancient royal pyramids over such a large territory may never been deduced. This article proposes a new theorem entitled "Royal Pyramids Linearity" (RPL) to introduce a common factor between the majorities of the ancient royal pyramids' locations in Egypt. The theorem is developed based on an assumption of the existence of linear connections between the ancient locations of some pyramids in Giza, Abusir, Saqqara and Dahshur. The theorem is proved mathematically through the construction of an optimization model that combined hypothesis testing and regression analysis. The model examined 43 royal pyramids. The results emphasized the existence of mathematical linear relationships of 34 that represent all the known royal pyramids constructed starting from the first true pyramid of King Senefru at the fourth dynasty till the last constructed pyramid at the eighteenth dynasty for King Ahmose excluding Khufu and Khafre pyramids. The theorem gives new explanations for the selection of Shepseskaf tomb and the pyramids of Userkaf, Sahure and Khentkaus. In addition, it provides new advantages for the locations of the tomb of Mentuhotep II and the pyramid of Khendjer.

KEYWORDS: Egyptian Royal Pyramids, Linear relationships, Mathematical model, RPL theorem, Zero-One Implicit Enumeration

1. INTRODUCTION

Egypt has a large number of discovered pyramids about 138 pyramids (Wood, 2020) classified into royal, subsidiary and satellite pyramids (Lehner, 2004).

The history of pyramids construction started from the reign of King Djoser from the third dynasty (2630–2611 BC) till the reign of King Ahmose from the eighteenth dynasty (1550–1525 BC) (Lehner, 2004). During that period there are 43 known royal pyramids. This article explores one of the factors used to specify the locations of the ancient royal pyramids.

The royal pyramids were not constructed within a specific site but they are distributed over a large area in Egypt starting from Abu Rawash at the north till Deir el-Bahari at the south (Fig. 1). Accordingly, there is a problem related to knowing the reason behind the distribution of the pyramids over such area.

The archaeologists proposed different factors that could have been used by the ancient Kings to locate their pyramids such as the pyramid should be located on the western bank of the Nile, close to the water (Barta, 2005), close to the capital, close to the king's palace, above a suitable level of the Nile, close to the stone quarries (Edwards, 1993), on the vision scope of the religious capital Heliopolis (Magli, 2010a), based on astronomical relationships by orienting the pyramid toward the true north (Lehner and Hawas, 2017) or based on local alignment with other ancient monuments (Lehner, 2004; Magli, 2010a; Magli, 2010b, 2011), but even structural properties e.g. measuring slopes, module divider, orientation and encoded coordinate system in the site-plan of the horizon of the Giza pyramids (Abdoulfotouh 2014, 2015), and geometric alignments (Clausen, 2016; Gonzales and Belmonte 2014). Some archaeologists concluded that we may never deduce the real factor leading to the distribution of the royal pyramids throughout the Egyptian territory (Barta, 2005).

This article proposes a new theorem entitled "Royal Pyramids Linearity" to find a common relationship between the majorities of the ancient royal pyramids locations. The theorem is developed based on an assumption of the existence of linear connection between the pyramid location with other pyramids. This assumption was discussed before for some of the Old Kingdom pyramids (Goedicke, 2001; Lehner, 2004; Dobrev, 2006, Magli, 2010a; Barta, 2012; Dickinson, 2014) and few of Middle Kingdom pyramids (Magli, 2012) while it was not proved mathematically before or studied for any pyramids from the First Intermediate Period, Second Intermediate Period or New Kingdom.

The importance of the theorem lies in finding new reason for the selection of the locations of the pyramids in some sites such as Saqqara, Abusir, Hawara, Lisht, El-Lahun and Dara, besides that it could be used to predict candidate locations of missing royal pyramids. Also, it provides new privileges that have never been addressed before for the locations of some pyramids such as the pyramid of Queen Khentkaus at Abusir, the tomb of Mentuhotep II at Deir el-Bahari and pyramid of Khendjer at Saqqara.

The article layout starts with discussing the assumption of the existence of linear relationships between the ancient pyramids, followed by proposing the RPL theorem, then proving the theorem by constructing a mathematical optimization model, followed by the discussion of the model results.

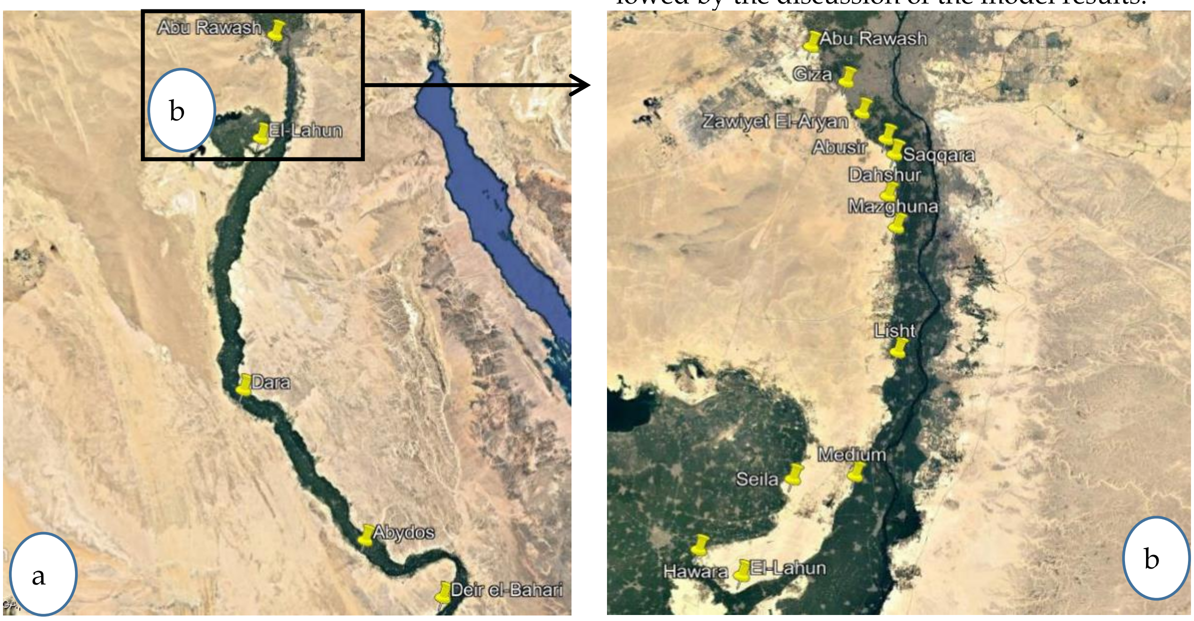

Figure 1. a: Distribution for the royal pyramids locations in Egypt; b: Pyramids locations between Abu Rawash and El-Lahun

2. THE LINEAR CONNECTION BETWEEN THE ANCIENT PYRAMIDS

Archaeologists have discussed the existence of linear relationships between the ancient monuments in different contexts such as the relation that connects the pyramids of the fourth dynasty in Giza plateau. This relation was formed through a line passing through the south eastern corners of Khufu, Khafre and Menkaure pyramids which is supposed to connect the pyramids with the core of the sun cult of Heliopolis (Goedicke, 2001; Lehner, 2004; Magli, 2009; Magli, 2010b; Barta, 2012; Waziry, 2020).

Other lines studied to connect the location of Userkaf pyramid in Saqqara from the fifth dynasty; one of the lines connected the pyramid with Khufu pyramid and the sun temple of Userkaf at Abusir which is supposed to create a religious connection between Userkaf and Khufu (Goedicke, 2001; Magli, 2010b). Besides a line connects the pyramid with the southern pyramid of King Senefru at Dahshur and the tomb of king Shepseskaf at Saqqara in order to make a symbolic ideological connection with the predecessor kings of Userkaf to be able to face the economic, religious and social challenges (Barta, 2012). Also, the location of the pyramid is supposed to be added at Saqqara in order to form a line known as "Saqqara Axis" which is passing roughly through the southeast corners of Djoser and Userkaf pyramids with the northwest corner of Sekhemkhet pyramid. This axis is extended at the end of the fifth dynasty to locate the pyramid of King Unas too (Magli, 2009; Magli, 2010b).

There are also other connecting lines for the pyramids of the sixth dynasty. The first line connects the pyramid of Teti with Djoser, Sekhemkhet, Userkaf and Unas pyramids (Magli, 2009; Magli, 2012). Other lines defined as the "rule of axis" such as a line connects the pyramid of King Pepi I at South Saqqara with Sekhemkhet and Djedkare Isesi pyramids, besides a line connecting the pyramid of King Merenre with Shepseskaf tomb and Unas pyramid. In addition to a line connecting the pyramid of King Pepi II with Djoser pyramid and the southern pyramid of King Senefru (Dobrev, 2006) which is interpreted as a way provided by King Pepi II to face the declining of the sixth dynasty by connecting the king with the icons of the previous dynasties of Djoser and Senefru (Barta, 2012). Other relationship is defined for the sixth dynasty called the "meridian axe" which is a straight line directed from the north to the south to align new constructed monument to the true north based on a reference of other preexisting pyramids (Magli, 2010b).

Few other linear relations were addressed before for the pyramids of the Middle Kingdom such as aligning the pyramid of Amenemhat II at Dahshur with the south base of the North pyramid of King Senefru, besides aligning the pyramid of Amenemhat II with the western side of the Bent pyramid at Dahshur (Magli, 2012).

From the previous studies, there are some drawbacks for the coverage of the existence of linear relationship between the pyramids such as: none of the previous studies proved the linear relationships mathematically to show the accuracy of the proposed lines. Also, all the previous studies are limited to the Old Kingdom pyramids besides few pyramids from the Middle Kingdom. In addition, they are limited to few number of royal necropolises of Giza, Abusir, Saqqara and Dahshur only. Also, there is no unification for the reference points included at the straight lines since the corners were used for some pyramids while the tops were used in other pyramids (Lehner, 2004; Magli, 2010b). Furthermore, the studied lines include other monuments rather than the royal pyramids such as the sun temple.

3. THE ROYAL PYRAMIDS LINEARITY THEOREM FOR LOCATING THE PYRAMIDS

The proposed Royal Pyramids Linearity theorem is aimed to provide comprehensive coverage for the existence of linear relationships among the royal pyramids during all the eras of pyramids construction of the Old Kingdom, First Intermediate period, Middle Kingdom, Second Intermediate period and New Kingdom, besides the coverage of pyramids in all the ancient royal necropolises in Egypt. In addition to that the theorem unifies the reference point of the pyramid to the tops with an exception of the rectangular tomb of King Shepseskaf which is represented by its four corners.

The Royal Pyramids Linearity theorem states that:

"Starting from the first true pyramid in ancient Egypt of the north pyramid of King Senefru from the fourth dynasty till the last constructed pyramid of King Ahmose at the eighteenth dynasty, all the royal pyramids except Khufu and Khafre pyramids are located on a direct linear relationship with at least two other pre-constructed pyramids' tops through at least one straight line".

The proof of the theorem required the construction of a mathematical optimization model to find the maximum number of straight lines passing through at least three pyramids tops putting into consideration the chronology of pyramids construction. The explanation of the model will be shown in the following section.

4. PROPOSED MATHEMATICAL OPTIMIZATION MODEL

4.1 Model Formulation

The mathematical optimization model is a model that seeks to find the maximum or minimum value of a specific function known as objective function, depending on a set of independent variables called decision variables while they may be related together through one or more constraints (Hillier et al., 2012).

The proposed model consists of an objective function represents the maximum sum of occurrence of all the possible lines that could be achieved given two other equations represent the constraints of the existence of linear relationships among the pyramids' tops.

The model inputs are the points (x_i, y_i) which represent the coordinates of the top of pyramid p_i, $p_i \epsilon P$ while P represents the set of all pyramids starting from the first constructed pyramid of King Djoser at the third dynasty till the last constructed pyramid in Egypt of King Ahmose at the eighteenth dynasty. The values of x_i and y_i are the longitude and latitude of the top of pyramid p_i, respectively.

The model decision variables are $X_{i,j,k}$ $\forall p_i, p_j, p_k \in P$. Each $X_{i,j,k}$ is a binary variable takes 1 if and only if there is a line passing through the tops of pyramids p_i,p_j and p_k, given that p_k was constructed before p_j, and p_j was constructed before pyramid p_i, while it takes 0 otherwise.

The objective function is the maximum available sum for all the possible straight lines passing through the tops of pyramids p_i,p_j and p_k, which is formulated as follows:

$$Max \sum_{\substack{\forall p_i, p_j, p_k \in P \\ i>j>k}} X_{i,j,k} \tag{1}$$

The model has two types of constraints aimed to provide restrictions for the existence of accurate linear relations between the pyramids tops.

The first type is formed based on the use of regression analysis to specify the best fit equation for the straight line passing through a set of points. The regression analysis is suitable in the model as it effective to predict the linear relationship between different variables such as the longitude and latitude of the pyramids locations.

In regression analysis the dispersion between the actual points and the fitted line is called the standard error of estimate (Lind et al., 2012).

The constraint is formed by calculating the standard error of the regression line passing through the tops of the pyramids p_i,p_j and p_k. The standard error is calculated according to the formula $\sqrt{\frac{\sum(y-\hat{y})^2}{n-2}}$ (Siegel, 2016). The value n represents the number of points included in the regression line which is constant to 3 since each line is passed through only three pyramids' tops. Also $\hat{y}$ is the predicted regression value at the point (x, y) (Lind et al., 2012).

In this case there are three predicted values of $\hat{y}_i$, $\hat{y}_j$ and $\hat{y}_k$ corresponding to the coordinates (x_i, y_i), (x_j, y_j) and (x_k, y_k) for tops of pyramids p_i,p_j and p_k respectively.

The value 0.001 is assumed as the maximum allowable standard error for the regression lines to return high accurate straight lines. In this way, the standard error is restricted by the formula $\sqrt{\sum(y-\hat{y})^2} \leq 0.001$.

The constraint is formulated by transferring all the terms included in the formula in one side then substitute by corresponding parameters for the coordinates and predicted values for the tops of pyramids p_i,p_j and p_k.

In this way, the first type of constraints is formulated as follows:

$$X_{i,j,k}\left(\sqrt{(y_i-\hat{y}_i)^2+(y_j-\hat{y}_j)^2+(y_k-\hat{y}_k)^2}-0.001\right) \leq 0 \tag{2}$$

The variable $X_{i,j,k}$ is multiplied by the constraint to take the value of 1 if and only if the value of the standard error is already less than 0.001.

The second type of constraints is aimed to validate statistically the results obtained from the first type of constraints.

The importance of this type comes from the use of a sample size of only three points in each fitted regression line so that the standard error may be close to zero while there is no actual linear relationship between the studied points (Render et al., 2012).

The statistical validation is conducted through the use of test of hypothesis to check the existence of linear relationship between the variables X and Y (Render et al., 2012). In this case X represents the longitude and Y represents the latitude variables. The hypothesis testing is suitable as it used to validate specific assumption related to the population based on data taken from a selected sample (Lind et al., 2012).

The general equation of the straight line is $Y = \beta_0 + \beta_1 X + \varepsilon$. This equation used to predict the relationship between two variables X and Y such as predicting the sales values based on the payroll. The parameter β_1 is the slope of the regression line, β_0 is Y-intercept and ε is an unpredicted random error. This equation is required to be tested based on the following hypotheses (Render et al., 2012):

$H_0: \beta_1 = 0$ "There is no linear relationship between X and Y"

$H_1: \beta_1 \neq 0$ "There is a linear relationship between X and Y"

The F distribution is employed to conduct the test (Lind et al., 2012; Render et al., 2012). The null hypothesis is rejected in case of the value of F statistic is greater than the value of F tabulated.

The value of the F statistic for the line passing through the tops of the pyramids p_i, p_j and p_k is calculated by dividing the value of mean square due regression $\left(\frac{\sum(\hat{Y}-\bar{Y})^2}{m}\right)$ over the mean squared error $\left(\frac{\sum(Y-\hat{Y})^2}{n-m-1}\right)$ while $\bar{y}$ is the mean value of y_i, y_j and y_k and m is the number of independent variables in the straight line equation which is 1 (Render et al., 2012).

The tabulated value of F distribution with significance level of 0.05 and degrees of freedom of $(m, n-m-1)$ is 161 (Lind et al., 2012).

In this way, the second type of constraint is formulated as follows:

$$X_{i,j,k}\left(\frac{(\hat{y}_i-\bar{y})^2+\left(\hat{y}_j-\bar{y}\right)^2+(\hat{y}_k-\bar{y})^2}{(y_i-\hat{y}_i)^2+\left(y_j-\hat{y}_j\right)^2+(y_k-\hat{y}_k)^2}-161\right)\geq 0 \quad (3)$$

The variable $X_{i,j,k}$ is multiplied by the constraint to take the value of 1 if and only if the value F statistic is greater than the value of F tabulated.

Based on that the proposed model is formulated as follows:

$$Max \sum_{\substack{\forall p_i,p_j,p_k\in P\\ i>j>k}} X_{i,j,k}$$

Subject to

$$X_{i,j,k}\left(\sqrt{(y_i-\hat{y}_i)^2+\left(y_j-\hat{y}_j\right)^2+(y_k-\hat{y}_k)^2}-0.001\right)\leq 0$$

$$X_{i,j,k}\left(\frac{(\hat{y}_i-\bar{y})^2+\left(\hat{y}_j-\bar{y}\right)^2+(\hat{y}_k-\bar{y})^2}{(y_i-\hat{y}_i)^2+\left(y_j-\hat{y}_j\right)^2+(y_k-\hat{y}_k)^2}-161\right)\geq 0$$

$$X_{i,j,k}\in\{0,1\}$$

4.2 Model Solution

The problem formulation includes integer binary variables in both the objective function and constraints which classifies under integer programing optimization problem. The Zero-One Implicit Enumeration Algorithm is effective in solving 0-1 integer optimization problems (Panneerselvam, 2016).

The algorithm enumerates all the possible solutions until reaching the optimal one. The enumeration process achieved through different iterations. In each iteration, a binary value is assigned to one of the variables then branch possible values for the other variables. For each valid combination of variables the objective function is updated until reaching the optimal value (Taha, 2014).

4.3 Model Implementation

The model implementation started with the input of the coordinates for all the discovered royal pyramids starting from first pyramid of King Djoser till the last pyramid of King Ahmose with a total of 43 pyramids. Each pyramid is assigned a unique index numbered sequentially from p_1 to p_{43} as shown in Table 1. The coordinates are collected using a geographic positioning system.

The model initializes all the values of $X_{i,j,k}\ \forall p_i, p_j, p_k \in P$ and i, j and k to zeros, then increases the value of i by 1 in each iteration. Starting from the value of i the model backward to assign decreased values for j and k. For each combination of i, j and k the variable $X_{i,j,k}$ is checked to satisfy to the constraints (2) and (3). In case of satisfying the constraints the variable $X_{i,j,k}$ is assigned the value of 1 and the value of objective function is updated. The previous steps are repeated till enumerating all the possible combinations.

The tomb of King Shepseskaf – with index p_{13} - from the fourth dynasty was considered as a challenge to the model implementation. This tomb has an ancient name of "The Purified Pyramid" (Lehner, 2004) while it was constructed as a rectangular tomb (Mastaba) instead of a pyramid (Siliotti and Hawass, 2003). The challenge here was how to specify the point(s) to represent the tomb in the model instead of top.

The challenge was overcome through the running of the model two times; the first run considered the tomb is represented by its central point with an index of p_{13}, while the second run represented the tomb through its corners with four additional indices of p_{13NE}, p_{13NW}, p_{13SE} and p_{13SW} for the northeast, northwest, southeast and southwest corners respectively.

4.4 Model Results

The first run of the model returned 251 straight lines (SLs) which increased to 318 in the second run covering 34 royal pyramids as shown in Table 1.

All the returned lines were studied while 247 lines are eliminated for different reasons such as: some pyramids are constructed too close to each other such as the pyramids (Neferirkare and Khentkaus) or (Shepseskaf and Pepi II) so that the same lines are repeated several times for different pyramids.

Also, other lines are eliminated because of the repeating of the same lines with other lines already developed for other pyramids.

The case of Neferirkare pyramid with index p_{17} includes the two types of eliminations since the model returned 21 lines connecting the pyramids from both the first and second runs while 17 were eliminated because of the closeness of Neferirkare pyramid with Khentkaus pyramid, besides that another elimination happened for 3 other repeating lines of (p_{14}, p_{15}),

(p_6, p_{16}) and (p_3, p_4) caused from previous returned relationships connecting the pyramids of Khentkaus and Sahure.

Table 1. Model results for the linear relationships among the Royal Pyramids

Pyramid	P. index	Location	Coordinates	Obtained SLs from first run	Obtained SLs from second run	Final SLs after elimination	Connected Pyramids
Djoser	p_1	Saqqara	29.871285, 31.216561	0	0	0	$\emptyset$
Sekhemkhet	p_2	Saqqara	29.866037, 31.212989	0	0	0	$\emptyset$
Khaba	p_3	Zawiyet El-Aryan	29.932853, 31.161208	0	0	0	$\emptyset$
Nebka	p_4	Zawiyet El-Aryan	29.940212, 31.151487	0	0	0	$\emptyset$
Meidum	p_5	Medium	29.388379, 31.157114	0	0	0	$\emptyset$
Seila	p_6	Seila	29.382608, 31.053533	0	0	0	$\emptyset$
Bent	p_7	Dahshur	29.790406, 31.209414	0	0	0	$\emptyset$
North	p_8	Dahshur	29.808700, 31.206206	1	1	1	(p_2, p_5)
Khufu	p_9	Giza	29.979295, 31.134192	0	0	0	$\emptyset$
Djedefre	p_{10}	Abu Rawash	30.032188, 31.074848	1	1	1	(p_2, p_4)
Khafre	p_{11}	Giza	29.976083, 31.130734	0	0	0	$\emptyset$
Menkaure	p_{12}	Giza	29.972569, 31.128250	1	1	1	(p_1, p_{10})
Shepseskaf	p_{13}	S. Saqqara	29.839004, 31.215203	1	3	2	$(p_3, p_9), (p_5, p_7)$
Userkaf	p_{14}	Saqqara	29.873457, 31.218833	1	2	1	(p_7, p_{13SW})
Khentkaus	p_{15}	Abusir	29.894204, 31.202372	1	1	1	(p_9, p_{14})
Sahure	p_{16}	Abusir	29.897723, 31.203286	2	2	2	$(p_6, p_{15}), (p_3, p_4)$
Neferirkare	p_{17}	Abusir	29.895097, 31.202364	17	21	1	(p_8, p_{15})
Shepseskare	p_{18}	Abusir	29.898799, 31.201311	10	18	1	(p_{13NW}, p_{15})
Neferefre	p_{19}	Abusir	29.893807, 31.201556	10	10	2	$(p_{11}, p_{14}), (p_8, p_{18})$
Niuserre	p_{20}	Abusir	29.895726, 31.203465	11	15	2	$(p_7, p_{16}), (p_{14}, p_{18})$
Headless	p_{21}	Saqqara	29.875276, 31.223592	16	16	3	$(p_{10}, p_{18}), (p_{12}, p_{20}), (p_{15}, p_{19})$
Djedkare Isesi	p_{22}	S. Saqqara	29.851061, 31.220800	3	3	2	$(p_{17}, p_{18}), (p_3, p_{11})$
Unas	p_{23}	Saqqara	29.868245, 31.214867	5	5	2	$(p_2, p_{14}), (p_3, p_{12})$
Teti	p_{24}	Saqqara	29.875251, 31.221723	18	18	3	$(p_9, p_{16}), (p_{11}, p_{20}), (p_{12}, p_{17})$
Pepi I	p_{25}	S. Saqqara	29.854495, 31.218884	17	21	3	$(p_5, p_{24}), (p_2, p_{22}), (p_{13}, p_{21})$
Merenre	p_{26}	S. Saqqara	29.850637, 31.215106	6	6	2	$(p_6, p_{21}), (p_4, p_{12})$
Pepi II	p_{27}	S. Saqqara	29.840334, 31.213506	11	25	4	$(p_{14}, p_{26}), (p_{10}, p_{13SW}), (p_8, p_{24}), (p_3, p_9)$
Qakare Ibi	p_{28}	S. Saqqara	29.841655, 31.217617	13	13	3	$(p_8, p_{22}), (p_{16}, p_{26}), (p_6, p_8)$
Khui	p_{29}	Dara	27.307838, 30.871608	3	3	1	(p_{24}, p_{25})
Mentuhotep II	p_{30}	Deir el-Bahari	25.737347, 32.606255	3	3	1	(p_{22}, p_{23})

Amenemhat I	p_{31}	Lisht	29.574917, 31.225373	3	5	3	$(p_7, p_{18}), (p_{13NE}, p_{26}), (p_{13SE}, p_{26})$
Senusret I	p_{32}	Lisht	29.560199, 31.221203	3	3	2	$(p_{16}, p_{20}), (p_{18}, p_{19})$
Amenemhat II	p_{33}	Dahshur	29.805836, 31.223128	8	16	2	$(p_{13SE}, p_{17}), (p_{13NW}, p_{19})$
Senusret II	p_{34}	El-Lahun	29.236281, 30.970614	7	9	3	$(p_{25}, p_{27}), (p_{16}, p_{17}), (p_{13SE}, p_{22})$
Senusret III	p_{35}	Dahshur	29.819047, 31.225559	11	14	2	$(p_{28}, p_{30}), (p_3, p_{27}), (p_{19}, p_{26})$
Amenemhat III	p_{36}	Dahshur	29.791885, 31.223731	8	10	2	$(p_{17}, p_{27}), (p_{13SW}, p_{16})$
Amenemhat III	p_{37}	Hawara	29.274275, 30.898867	8	10	2	$(p_{17}, p_{19}), (p_1, p_{23})$
Amenemhat IV	p_{38}	Mazghuna	29.761801, 31.220939	5	5	1	(p_7, p_9)
Sobekneferu	p_{39}	Mazghuna	29.767675, 31.220909	5	5	3	$(p_3, p_8), (p_{25}, p_{31}), (p_2, p_{13})$
Ameny Qemau	p_{40}	S. Saqqara	29.781934, 31.221498	4	4	1	(p_8, p_{10})
Unknown	p_{41}	S. Saqqara	29.830682, 31.222375	13	17	4	$(p_2, p_{35}), (p_{16}, p_{35}), (p_{13}, p_{27}), (p_1, p_{25})$
Khendjer	p_{42}	S. Saqqara	29.832469, 31.223875	17	20	5	$(p_{14}, p_{35}), (p_{13}, p_{27}), (p_2, p_{16}), (p_8, p_{41}), (p_{12}, p_{28})$
Ahmose I	p_{43}	Abydos	26.175019, 31.937834	8	12	4	$(p_{23}, p_{41}), (p_{16}, p_{20}), (p_{25}, p_{35}), (p_{13SW}, p_{16})$
				251	318	73	

The final number of returned straight lines after elimination is 73 lines covering 34 royal pyramids. The final results proved mathematically the Royal Pyramids Linearity theorem for the existence of linear relationship connecting the majority of 79.1% of all the discovered Egyptian royal pyramids. Besides that the theorem covered all the eras of pyramids construction starting from the Old Kingdom till the New Kingdom. In addition to that it covered pyramids in all the royal necropolises starting from Abo Rawash at the north till Deir el-Bahari at the south.

5. DISCUSSION

The model proved the RPL theorem for the existence of linear connection for 34 out of 43 pyramids location. There is no evidence available to know the ancient technique used for developing the linear connection among the pyramids. The 'imperishable' stars may be used to obtain the linear alignment among the ancient monuments (Magli, 2010b). In fact, there is a need for further studies to know how the ancient Egyptians were able to develop too long straight lines passing over most of the Egyptian territory.

The reason behind connecting the pyramids with each other may refer to religious or political purposes such as connecting the pyramid with the sacred places during the time of construction or giving the King symbolical connection with the icons of his predecessors (Barta, 2012).

The model results did not validate the linear connection for only nine pyramids. Seven out of the nine (Djoser to Bent) were constructed before the first true pyramid in ancient Egypt of King Senefru. The reason behind that may be for not reaching the final design of the pyramid since the shape of the pyramids changed from step to layer and bent (Siliotti and Hawass, 2003; Lehner, 2004), thus most probably the logic of the choice of the pyramid location may had not been finalized too.

The other two pyramids are Khufu and Khafre pyramids which may not abide to connection with other pyramids because these pyramids used astronomical logic to specify their locations as they have the most accurate alignment to the true north (Edwards, 1993; Belmonte, 2001; Lehner, 2004; Seyfzadeh, 2018; Ghosh, 2020). In fact, there is some difficulty to develop an optimization model covering the location of the nine pyramids because of the difficulty in finding common relationship between their locations.

The model results showed that the first applied pyramid for the theorem is the North pyramid of King Senefru at Dahshur which is considered as the first complete pyramid in ancient Egypt (Hemeda et al., 2019). The theorem adds another feature for the pyramid as it is the first pyramid located between the tops of two pyramids of Sekhemkhet and Meidum through a straight line with a total length of about 53.2 km.

The RPL theorem provides also new explanations for the locations of some pyramids besides it gives novel advantages for other pyramids. Some of these are the tomb of Shepseskaf and the pyramids of Userkaf, Sahure, Khentkaus and Khendjer.

The tomb of Shepseskaf was constructed during the fourth dynasty at South Saqqara. The RPL theorem provides solutions for two problems related to it;

there is no satisfactory explanation for the choice of the specific location of this Mastaba at the south of Saqqara and why this King chose to build a rectangular tomb instead of a pyramid like his predecessors (Barta, 2005). The solution depends on the impact created for the model results based on the changing of input of points related to the tomb. Since the tomb is presented in the model in two different formats of using a center point for the tomb at the first run while using its four corners at the second run. In the first run the model returned only one linear relationship to connect the center point to the two pyramids of Khaba and Khufu (p_3, p_9) with a standard error of 2.7× 10^{-4} while in the second run the model returned the connection of the same pair of pyramids with the north east corner of the tomb with less standard error of 3.2×10^{-5} in addition to appearance of another relationship connecting the south east corner with Meidum and Bent pyramids (p_5, p_7). The reduction of the standard error and the appearance of the other relationship at the second run give the indication of the construction of this rectangular tomb to connect at least four pyramids through the tomb's corners instead of its central point while the selection of South Saqqara may be due to the limitation of the locations availability to connect at least four pyramids during the time of construction.

The RPL theorem could answer some questions related to the locations of the pyramids of the fifth dynasty such as giving explanation for breaking of King Userkaf for the traditions from his predecessors who built their pyramids at Giza plateau and he built his pyramid close to Djoser complex at Saqqara (Siliotti and Hawass, 2003). The theorem showed that the pyramid of Userkaf has an advantage of locating it on the intersection of two lines containing four pyramids of (Bent, Shepseskaf) from the south and (Khufu, Khentkaus) from the north. The first line already studied before (Dobrev, 2006; Barta, 2012) while the second line is addressed for the first time and it may reflect an important connection between King Userkaf and Queen Khentkaus through her pyramid at Abusir.

The theorem also provides a new advantage for the location of pyramid of Sahure at Abusir since there is no clear reason for establishing this pyramid at Abusir (Barta, 2012). RPL theorem showed that the location of pyramid was chosen carefully on the intersection of two straight lines including four pyramids of (Selia, Khentkaus) and (Khaba, Nebka).

One of the important results of the RPL theorem is the showing of a unique advantage for the location of the pyramid of Queen Khentkaus at Abusir as it is considered a pivot pyramid for the pyramids of the first four kings of the fifth dynasty of Userkaf, Sahure, Neferirkare and Shepseskare since it is located on the intersections between four pairs of pyramids of (Khufu, Userkaf), (Seila, Sahure), (North, Neferirkare) and (Shepseskaf, Shepsekare) as shown in Figure 2. This location may support the historical role of Queen Khentkaus during the fifth dynasty.

The model results emphasized some relations proposed for the sixth dynasty for locating the pyramids at south Saqqara under the rule of axis (Dobrev, 2006) such as the line connecting the pyramid of King Pepi I with Sekhemkhet and Djedkare Isesi (p_2, p_{22}) while the results produced additional relations connecting the same pyramid of (Meidum, Teti) and (Shepseskaf, headless).

The theorem also showed other privileges for the tomb of the king Mentuhotep Neb Hebt Re (Mentuhotep II) from the Middle Kingdom at Deir el-Bahari in Upper Egypt. This tomb described as a pyramid in some of the ancient texts such as Abbott Papyrus. The pyramid is currently collapsed while it was built at the center of the edifice (Lehner, 2004). The pyramid's top added on one of the longest straight lines known in the ancient history which started at the top of Unas pyramid at Saqqara, passes through the top of Djedkare Isesi pyramid and ended at the tomb of Mentuhotep II with a total length about 477.9 km as shown in Fig. 3.

The RPL theorem addressed also a unique advantage for the pyramid of Khendjer from the Second Intermediate Period at South Saqqara. The pyramid is located on the intersection of five lines containing ten pyramids (Table 1) which is the maximum number of connecting pyramids achieved in ancient Egypt.

Finally, The Royal Pyramids Linearity theorem considered as an extensive work for use of mathematics in pyramids construction (Vafea, 2002; Imhausen, 2020) by defining new mathematical relationships used by the ancient Egyptians to specify the locations of the pyramids.

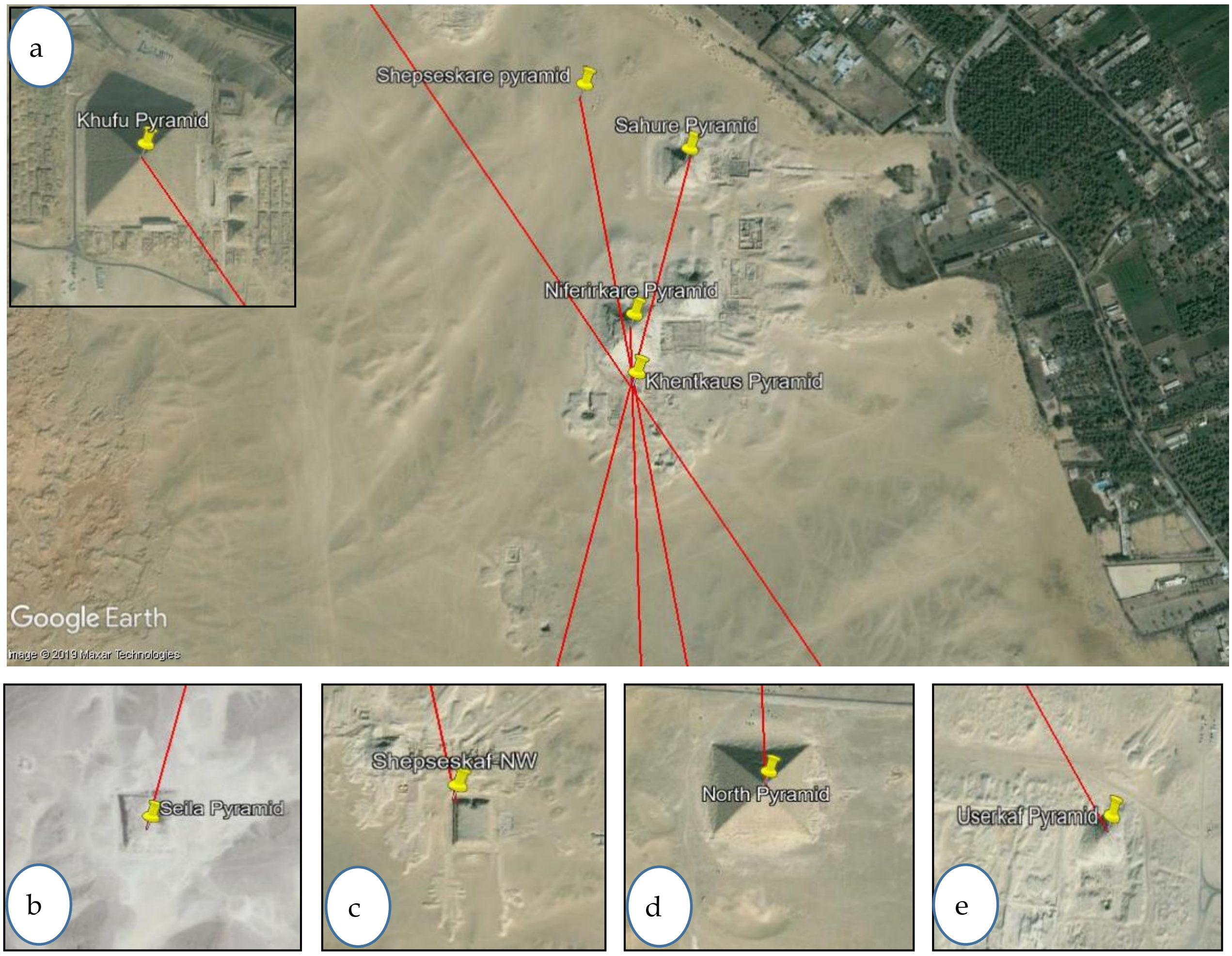

Figure 2. Locating the pyramid of Khentkaus at Abusir with a direct connection with the pyramids of Userkaf, Sahure, Neferirkare and Shepseskare through the lines (Khufu – Userkaf), (Seila, Sahure), (North pyramid of Senefru – Neferirkare) and (Shepseskaf tomb (NW) – Shepseskare) respectively; a: The pyramid of King Khufu at Giza; b: The pyramid of Seila for King Senefru at Fayoum; c: The tomb of King Shepseskaf at Saqqara; d: The North pyramid of King Senefru at Dahshur; e: The pyramid of King Userkaf at Saqqara.

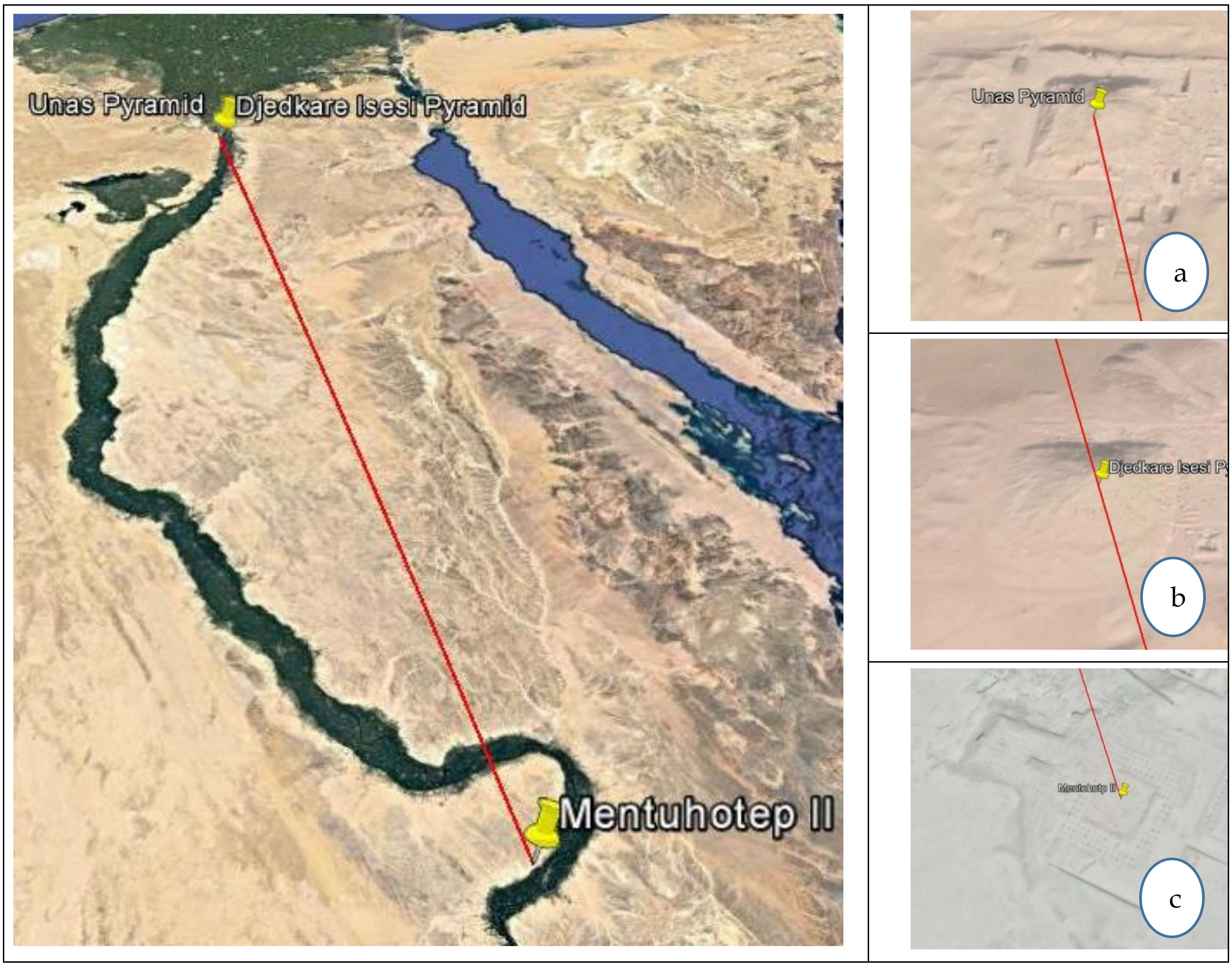

Figure 3. The longest straight known in ancient Egypt connecting the monetary temple of Mentuhotep II tomb with tops of Unas and Djedkare Isesi pyramids; a: The pyramid of King Unas at Saqqara; b: The pyramid of King Djedkare Isesi at South Saqqara; c: The monetary temple of Mentuhotep II tomb at Deir el-Bahari.

6. CONCLUSION

The article proposes a new theorem of Royal Pyramids Linearity to solve some of the problems related to the royal pyramids locations. This theorem showed that each royal pyramid should be located on a direct linear relationship with at least two other pyramids through at least one straight line. The theorem proved by developing of a mathematical optimization model solved by Zero-One implicit enumeration technique. The model results emphasized the theorem concept over 79.1% royal pyramids including all the constructed pyramids starting from the first true pyramid in ancient Egypt of King Senefru from the fourth dynasty till the last constructed pyramid of King Ahmose from the eighteenth with only two exceptional cases of Khufu and Khafre pyramids.

The RPL theorem solved some of the problems related to the ancient pyramids and tombs such as it showed new advantage for the building of King Shepseskaf to a rectangular tomb instead of a pyramid to allow the tomb to be connected with four other pyramids through its corners. Also it solved the problems related the pyramids of Userkaf and Sahure while both of them are added carefully in a location where four other pyramids are intersected together through two straight lines. The theorem illustrated also a new advantage never addressed before related to the pyramid of Queen Khentkaus at Abusir which is the only location that connects the pyramids of the first four Kings of the fifth dynasty of Userkaf, Sahure, Neferirkare and Shepseskare together to reflect her important role during the fifth dynasty. In addition, the theorem showed new privilege for the location of Mentuhotep II tomb at Deir el-Bahari as it located on one of the longest straight lines in the ancient history of a total length of about 477.9 km passing through Unas and Djedkare Isesi pyramids, besides another advantage for Khendjer pyramid as it located where ten pyramids are intersected together through five straight lines which is the maximum number of connections achieved during the ancient time. The Royal Pyramids Linearity theorem could be effective in the future in finding new candidate locations of undiscovered royal pyramids or tombs with the reduction of the money and time required for excavations.

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