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HIGHLY ADVANCED FOR THE ERA GEOMETRIC STENCILS WERE USED FOR THE DRAWING OF CELEBRATED MINOAN LATE BRONZE AGE WALL-PAINTINGS: NEW DATA

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ABSTRACT

In this work, the method of drawing of nine celebrated Minoan Late Bronze Age wall-paintings, is studied. Four of these frescoes were unearthed at Akrotiri, Thera, while the other five were excavated in Crete; these frescoes have never been thoroughly studied before, concerning their method of drawing. The authors demonstrate that practically all actually drawn contours in these frescoes, optimally fit six geometric stencils-guides, namely, four hyperbolae and two Archimedes' (linear) spirals. It is important to note that the hyperbola and the linear spiral do not exist in nature, definitely not with the precision encountered here. We would like to stress that all wall-paintings the authors have studied so far, i.e. more than twenty-five (25), had been drawn via the use of the same stencils, a fact that renders the eventuality that this was indeed the method of drawing of these frescoes, almost certain. The considered wall-paintings cover a period of at least two centuries, located both before and after the gigantic eruption of Thera volcano. Moreover, here, possible methods of construction of the determined stencils are proposed, as well as the most probable one, according to our opinion. Methodologically, the authors determine couples of the longest contour parts, CP, appearing in the studied frescoes and of a proper sub-curve GP^{opt} of one of the aforementioned stencils; these couples are chosen so as they fit with exceptionally low approximation errors. The latter terms are rigorously defined here, for the first time. Special care is taken, mathematically and programmatically, to ensure the maximum possible smoothness between successive optimally fit stencils' parts GP^{opt}. Next, the authors employed these modern prerequisites, to introduce the more probable sequence of actions taken by the prehistoric artist(s) for drawing the specific frescoes. Finally, crucial queries and conjectures are set, together with a number of plausible answers.

KEYWORDS: Prehistoric Wall-Paintings, Frescoes' Method of Construction, Late Bronze Age Art, Geometry in Prehistoric Aegean, Stencils, Linear Spiral, Archimedes' spiral, Hyperbola

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1. INTRODUCTION

1.1. *The civilization that flourished at Akrotiri, Thera in the Late Bronze Age, with emphasis to the wall-paintings excavated there*

According to numerous archaeological observations and findings, the Cycladic Islands had been inhabited from the 8th millennium B.C. (e.g. (Doumas, 1992; Sakellarakēs and Ntumas, 1994; *The Oxford Handbook of the Bronze Age Aegean*, 2012)). However, the present work focuses on four (4) wall-paintings that belong to the marvellous civilization, which flourished at Akrotiri, in the particularly beautiful island of Thera, in the Late Bronze Age. In essence, the authors of the work in hand, continue to study the method of drawing of the frescoes unearthed at Akrotiri, which most probably had been made somewhere in the second half of the 17th century B. C. A number of researchers-archaeologists refer to this time period as "the Middle Cycladic period, that extends from 2000 B.C. to 1600 B.C." (e.g. (Doumas, 1992; Sakellarakēs and Ntumas, 1994)); however, in the present manuscript, the authors will try to use these types of terminology as little as possible, since it seems that there is a considerable discussion/scientific dispute concerning the exact limits of the corresponding periods (e.g. (Friedrich et al., 2006; Warren and Hankey, 1989)).

One may make a number of quite plausible remarks concerning the inhabitants of Cyclades (e.g. (Sakellarakēs and Ntumas, 1994)). In fact, the rather serious isolation of these inhabitants from the mainland, together with the strong wind that frequently prevails in these islands, the lack of rivers and rich springs, the limited rainfall, the relatively small cultivable area etc., forced the inhabitants of Cyclades to develop their inventiveness and creativity ((Doumas, 1992; Sakellarakēs and Ntumas, 1994)). Hence, many of these inhabitants, like those of Akrotiri, Thera, accumulated an extraordinary for the era amount of empirical knowledge, concerning navigation, Naval Architecture and Construction, Hydrodynamics, Aerodynamics, Architecture and Building etc. In turn, navigation skills required a good empirical knowledge of astronomy, of the behaviour of sea-currents, an astute observation of the environment, etc. (Doumas, 1992; Sakellarakēs and Ntumas, 1994).

Furthermore, we shall show here that the inhabitants of Akrotiri (at least?) had acquired an exceptional and impressive empirical knowledge of Geometry and Technology. Indeed, we shall extend and reinforce analogous opinions expressed in (Papaodysseus et al., 2005) by demonstrating that the "subconscious-emotional knowledge" of Geometry and Mathematics of Classical Ages, definitely existed in the Aegean islands. We shall prove these claims by establishing the method of drawing of celebrated wall-paintings unearthed at Akrotiri, Thera and in Crete (see Sections 4, 5 and the Supplementary Material).

1.2. *A Brief Reference to the Art of the Wall-Painting in the Minoan Civilization*

According to all findings, in the periods from 2000 B.C. to 1600 B.C. in one hand, and from 1600 B.C. to 1450 B.C. on the other, there was the heyday of the Minoan Art of painting ((Sakellarakēs and Ntumas, 1994; *The Oxford Handbook of the Bronze Age Aegean*, 2012)). In particular, concerning the wall-paintings, we should emphasize that the Minoans respected human dimensions and measures in them. This artistic attitude was in full contrast with the corresponding behaviour of other great civilizations contemporary to the Minoan one (e.g., the Egyptians), which "adored" the gigantic forms and constructions. In fact, the members of these civilizations had to deal with "huge" areas with respect to the human size; thus, in order to balance this contrast, they created gigantic sculptures, pyramids, etc. On the contrary, the Minoan civilization had flourished in relatively pretty small islands and the Minoans had reconciled with these limited areas. It seems that the latter had influenced the Art of Minoan wall-painting. In fact, in most unearthed Minoan frescoes, the artist(s) had respected human dimensions and scale ((Doumas, 1992; Sakellarakēs and Ntumas, 1994)). In addition, clearly, the dimensions are smaller or even impressively smaller in the artifacts of seal engraving or glyptic as well as in various products of the goldsmith's work. Consequently, "Minoan art may be succinctly described as the art of the miniature" ((Sakellarakēs and Ntumas, 1994)). Furthermore, a number of researchers has noticed repetitions in various classes of Minoan wall-paintings (see Section S.M.1 of the Supplementary Material and (Bietak Manfred et al., 2007; Crowley, 1997)).

1.3. *The Novel Aspects of the Present Work*

1. We demonstrate that nine (9) additional wall-paintings (described in Sections 2, 4, 5 and S.M.2 and S.M.3 of the Supplementary Material), the method of construction of which has never been studied before, were, very likely, drawn by the stencils presented in Section 2.
   We feel obliged to emphasize that repetition of an event, systematically, implies causality/the existence of a law; we believe that this is the essence of science. Thus, if one also considers the nine frescoes or frescoes' fragments studied here, brush-contours of more than 23 cm long or even

of length greater than 27 or 30 cm fit the very same geometric curve among those of Section 2, which we have determined uniquely, with an average approximation error less than 0.3-0.4 mm (millimetres) and a maximum one smaller than 0.6-0.8 mm (millimetres). This fact has been observed in more than two hundred (200) instances, appearing in at least twenty (20) wall-paintings or large fragments of them. The probability that this is not accidental, has been increased by the nine frescoes referred to in the present work, so that one may consider it to be one. Equivalently, the study of the nine additional wall-paintings introduced here, renders the eventuality that all frescoes studied so far had been drawn by the method presented by the authors, is practically certain.
2. Moreover, the more probable method the prehistoric artist(s) used for drawing these frescoes is introduced in the present work. In fact, we have (we believe strongly) supported our hypothesis that all Late Bronze Age frescoes excavated at Akrotiri, Thera and Crete, which we have studied so far, had been drawn by a method pretty analogous to the one introduced here.
3. We, for the first time, give a full exposition of probable methods of construction of the stencils employed by the prehistoric artist(s). We have taken special care to ensure that the proposed methods of construction are compatible with a specific era.
4. We have explicitly given the rigorous definition of the error with which a stencil part optimally approximates a corresponding segment of the contour of a shape appearing in a fresco. In particular, we have given a rigorous definition of the "minimum (mean) error" and of the associated "maximum error" in subsection 3.2.
5. Using these approximation errors, we have reported the methodology we have applied, for testing our hypothesis, in a stricter manner.
6. We have tried to give special emphasis to the evolution of the mathematical thought in the Hellenic Region and to reveal that the (probably empirical, sub-conscious and emotional) origins of it, are in the Late Bronze Age Aegean. We would like, once more, to emphasize that linear spirals and hyperbolae do not exist in nature and definitely not with the precision encountered in the contours of the various wall-paintings figures.
7. Analogous statements hold true for the empirical knowledge of Technology in the Aegean Sea, but not only, and its evolution throughout the centuries in the context of the Minoan and other civilizations that flourished in the islands and the shores of Aegean Sea.
8. In the last Section 6, we have set a number of questions and conjectures we believe they are crucial, to which we have tried to give plausible answers.

2. THE BASIC CONJECTURE STATED AND DEMONSTRATED BY THE AUTHORS OF THE PRESENT WORK

The authors, here, will analyse and, according to their opinion, will establish the method of drawing of two sets of very important wall-paintings: the first set has been unearthed at Akrotiri, Thera and the other one in Crete. In particular, the authors have dealt with the following two groups of frescoes:

A. Frescoes excavated at Akrotiri

A1. the figure of "The griffin" belonging to the synthesis "The Crocus gathering".

A2. the fresco "Sea Daffodils" or "Lilies".

A3. the middle figure of the synthesis "Adorants".

A4. the wall-painting named "Fisherman".

A brief description of each one of these frescoes will be given in Section 4.

B. A group of three (3) wall-paintings belonging to the Minoan Crete civilization:

B1. the wall-painting "the Prince of Lilies".

B2. the wall-painting "The Cup-Bearer" or "The Rhyton-Bearer".

B3. the Minoan fresco "The blue bird".

In essence, this second group comprises five (5) wall-paintings, given that, most probably, the fresco "the Prince of Lilies" is an assemblage of three (3) fragments belonging to three different syntheses, as we will describe below. Again, a short presentation of each one of these frescoes will be given in Section 5.

The basic conjecture associated with the present work is an extension of the ones introduced in (Papaodysseus et al., 2022, 2008, 2006a, 2005). However, we must emphasize that the frescoes the authors deal with in the present work, have never been studied before. We note that the Minoan wall-paintings studied so far, which verify this conjecture, extend to a large time period from 1700 to 1400 B.C. (see Section 5).

More specifically, here, the authors will further support the validity of the following "strong" conjecture: all the aforementioned wall-paintings had been, practically exclusively, drawn by means of a specific, small ensemble of highly advanced for the era, geometric prototypes, which are explicitly presented in the following subsection 2.1.

This fact completely answers the existence and the appearance of the repetitions observed by other researchers, as it is referred to in the Introduction and in the Supplementary Material (S.M.1). It also fully explains the stable and clear-cut border lines that characterize all these frescoes, especially since this stability should have been achieved very fast, given that the drawing surface (i. e. the plaster) was wet.

In addition, one cannot exclude the existence of prefabricated templates that played the role of prototypes/ "pole stars" for various figures appearing in wall-paintings.

Furthermore, in the manuscript in hand, we shall try to convey our opinion concerning the highly advanced for the era subconscious (at least) and early - protogenic knowledge of Geometry and Technology of the inhabitants of Thera, Crete and other Aegean islands.

2.1. *The Geometric Guides that Have Been Used for the Drawing of the Aforementioned Wall-Paintings*

The basic hypothesis that will be tested and confirmed in the present work, is that all contours of the previously mentioned large set of prehistoric wall-paintings, which have been unearthed at Akrotiri, Thera and in Crete, had been very probably drawn by means of a very limited set of geometric guides. Previous relative studies concerning a number of celebrated wall-paintings of the Late Bronze Age have already been published in (Papaodysseus et al., 2022, 2006a, 2006b, 2005); the corresponding publications mainly deal with the aspects associated with the scientific disciplines of Mathematics and Computer Engineering. These studies are substantially extended and improved here, in connection with the aforementioned set of Minoan wall-paintings.

Conclusively, the main goal of the present work is to determine-reconfirm the minimum number and the type of geometric guides that had been used in the drawing of the largest possible set of frescoes in the Aegean Sea during the Late Bronze Era. Evidently, first, one must prove the compatibility of such a group of guides with the actually drawn border lines of the studied frescoes. In fact, the authors have demonstrated and manifested here, for the first time, that nine (9) wall-paintings and/or fragments of them include contour lines, which, practically all, perfectly match an absolutely minimum set of geometric guides (as it has already been mentioned before four (4) excavated at Akrotiri, Thera and five (5) belonging to the Minoan Crete civilization). Indeed, this set consists of four (4) hyperbolae and two (2) linear/Archimedes' spirals, which were highly advanced for the Late Bronze Age and, of course, for any previous prehistoric era. The equations of these geometric guides are stated below, while their shape is depicted in Figures 1 and 2.

2.1.1. The General Equation of any Hyperbola-Guide

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \quad (1)$$

where x is a real independent variable, $x \in [x1, x2]$. The entire analysis performed so far, indicates that a very good choice for the independent variable x is $x \in [-30,30]$, with x and its limits expressed in cm. This choice does not imply that the artist(s) in the Late Bronze Age have used the entire stencils introduced below, but they have certainly used a part of them in order to draw all the wall-paintings treated so far. Evidently, the criteria employed by those artist(s) were aesthetic ones.

The exact shape of a hyperbola depends on the precise value of the parameters a and b of (1). Four (4) different pairs of the parameter values (a, b) give rise to the four spotted stencils; these pairs are explicitly presented in Table 1. A unique color has been attributed to each such stencil-hyperbola, shown in Fig. 1 and Table 1.

At this point, we would like to emphasize that the values of the parameters a and b referred to in Table 1, give rise to exactly the same hyperbola eccentricity with the corresponding parameter values introduced in (Papaodysseus et al., 2006a, 2005).

Table 1. The exact parameters a and b, in cm, of the four hyperbolae prototypes determined by the authors.

Hyperbolae stencils	a (cm)	b (cm)
cyan	6.3551	7.6448
green	12.4854	19.1041
blue	23.8560	53.5219
Magenta	43.2391	61.0797

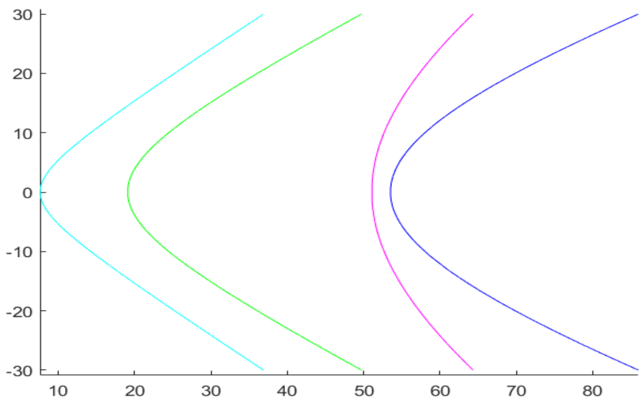


Figure 1. The four hyperbolae geometric prototypes. We stress that each hyperbola color is confined to and preserved for the corresponding geometric prototype and the associated guide only.

2.1.2. General Equation of the Archimedes'/Linear Spirals

$$\vec{r}(\theta) = k \cdot \theta \cdot \cos(\theta)\hat{\imath} + k \cdot \theta \cdot \sin(\theta)\hat{\jmath}, \ \theta \in [0,6\pi] \quad (2)$$

where θ is the polar angle. In essence, the Archimedes' spiral is a circle the radius of which linearly grows as a function of θ; for this reason, it is also called "a linear spiral". We stress that each distinct value of the parameter k gives rise to a different shape of the Archimedes' spiral, without, of course, changing its functional form. The two (2) distinct guides corresponding to two different values of the constant k, are depicted in Fig. 2 and they are shown in Table 2; we use the symbols "ls0" and "ls2" for these two linear spirals.

We, once more, would like to emphasize that we have evaluated a slightly improved value of the constant k in comparison with the ones presented in most of the previous studies. It goes without saying that the authors have reconsidered all the wall-paintings treated so far and they have found out a slightly better matching of the corresponding prototype curves to all actually drawn contours, in comparison with the values referred to in the previous publications (for a more rigorous related statement, see subsection 3.2). Of course, the essence of the approach and the power of the conjecture did not alter at all, but on the contrary, they were strengthened.

*Table 2. The exact parameters **k**, in cm, of the two Archimedes' spirals defined above.*

Archimedes' spiral stencils	***k***(cm)
ls0 (red)	0.0083
ls2 (ruby red)	0.113

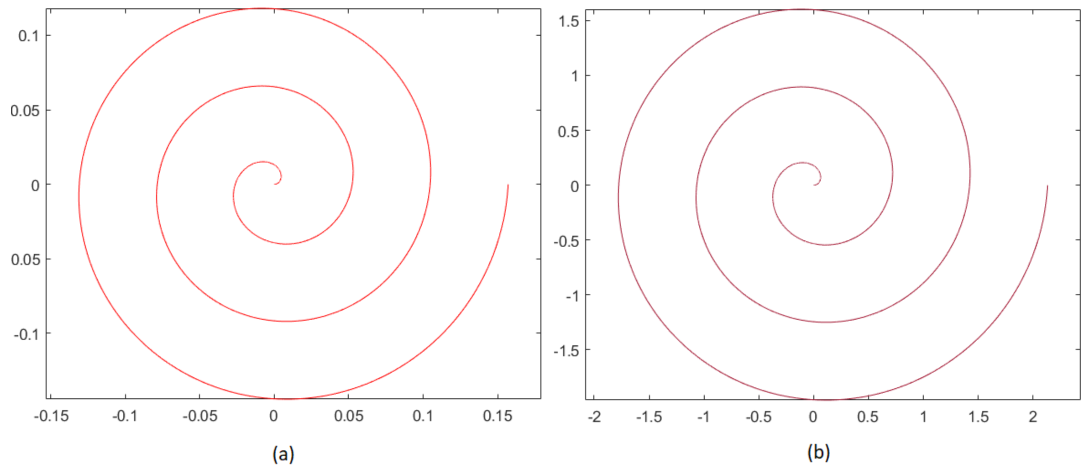


Figure 2. In sub-figure (a) Archimedes' spiral ls0 is depicted, always in red color, while in (b) Archimedes' spiral ls2 is depicted, always in ruby red color, introduced in Table 2. We stress that each spiral color is, again, confined to and preserved for the corresponding geometric prototype only.

2.2. *A Summary of the History of the Classical Ages Geometry and Mathematics, Associated with the Detected Guides*

In this subsection, a brief description of the knowledge in Classical Ages will be given, concerning geometry and mathematics; this description will mainly concentrate on geometric configurations that have been spotted by the authors in the considered prehistoric Aegean wall-paintings (Dantzig, 2006; Glaeser et al., 2018; Heath, 2013; Spandagos et al., 2000). In fact:

Concerning linear spirals, their definition is given in subsection 2.1; loosely speaking, a linear spiral may be considered to be a circle whose radius linearly increases with the corresponding polar angle. We note that this type of spiral does not exist in nature. Thus, its appearance on the prehistoric Aegean wall-paintings that is demonstrated in the present work, too, is a result of an impressively difficult invention/inspiration.

We emphasize that the columnar 3-D helix attested in the Old Babylonian civilization, frequently but erroneously called a "linear spiral", at first is a three-dimensional curve, with a completely different functional form and therefore has nothing to do with the linear spiral that appears in the considered Aegean wall-paintings (Robson, 1999). Moreover, the Babylonian 3-D helix is extremely more easily generated than the Archimedes' spiral.

So far, the conception and the mathematical definition of the "linear spiral" has been attributed to Conon of Samos (Κόνων ο Σάμιος), a friend of the great Archimedes. Consequently, it is not a surprise that Archimedes (Αρχιμήδης ο Συρακόσιος) in his celebrated book "On Spirals" ("Περί Ελίκων") gave a rigorous definition of the "linear spiral", together with a considerable number of theorems concerning properties of this geometric scheme (Archimedes and Heath, 2009). In particular, it is worthwhile noticing that Archimedes made the impressive achievement to both trisect an angle, as well as to square the circle, by using the linear spiral. For this reason, this spiral till nowadays bears his name.

Concerning conics, so far, it is believed that the first who conceived them and realized that they result from the intersection of a cone with a plane was Menaechmus (Μέναιχμος εκ Προκοννήσου), around 350 B.C. It seems that Menaechmus dealt with conics in his effort to solve the celebrated "Delian Problem" ("Δήλιον Πρόβλημα"), namely the doubling of the volume of a cube, using compasses and rulers exclusively. The first who wrote about conics is Euclid (Ευκλείδης) around 300 B.C. According to Pappus (320 A.D.), "the four books of Euclid's Conics, were completed by Apollonius (Απολλώνιος ο Περιεύς), who added four more books of Conics". In fact, "the great Geometrician" Apollonius continued and extended the work of Menaechmus: many complicated theorems concerning conics have been stated and proved by Apollonius, while the names of the three conics' types (ellipse, hyperbola, parabola) are attributed to him.

It would be unfair not to mention emphatically, that the works of the aforementioned "giants" of mathematical thought had been influenced by the studies of Thales of Miletus ("Θαλή του Μιλησίου", 640 π.Χ. - 546 π.Χ) and of Pythagoras of Samos ("Πυθαγόρα του Σάμιου", 580 π.Χ. -, 496 π.Χ.). Actually, as we shall state in the next subsection 2.3, probable methods of construction of the Geometric guides presented in subsection 2.1, may be associated with the geometric results of Thales and Pythagoras that appear many centuries later.

Thus, once more, we shall emphasize that the conception and construction of conical sections requires a considerable amount of inspiration and technical skills (Besant, 2016; Glaeser et al., 2018).

2.3. A Number of Plausible for the Era Methods of Construction of Hyperbolae and Linear Spirals

After a rather extensive search, we have concluded that certain methods of construction must be excluded for various reasons, which will be described below. On the contrary, two (2) such methods seem probable or, at least, not immediately rejectable. To the best of our understanding, one method is the most probable candidate for being used in the Late Bronze Age. A brief corresponding analysis follows:

2.3.1. MC1. Eventuality of exploiting the interference fringes

In contemporary physics, one may be tempted to consider that the interference fringes may be the basis for constructing a hyperbola. In addition, one may guess that the interference phenomenon may lead to encounter hyperbola in everyday life. Both these considerations are completely wrong; this is due to the fact that the interference fringes are highly noisy as Fig. 3 clearly and undoubtedly manifests. In fact, as we shall demonstrate throughout the present work, the employed geometric guides in the Late Bronze Age were generated with an impressive precision. Indeed, all realizations of these geometric prototypes differ in less than 0.2 mm in average from the corresponding contemporary curve, like the ones made by a modern computing machine.

The previous observations fully exclude the eventuality that the Aegean technologists, technicians and artists have employed the interference phenomenon to generate an exact hyperbola.

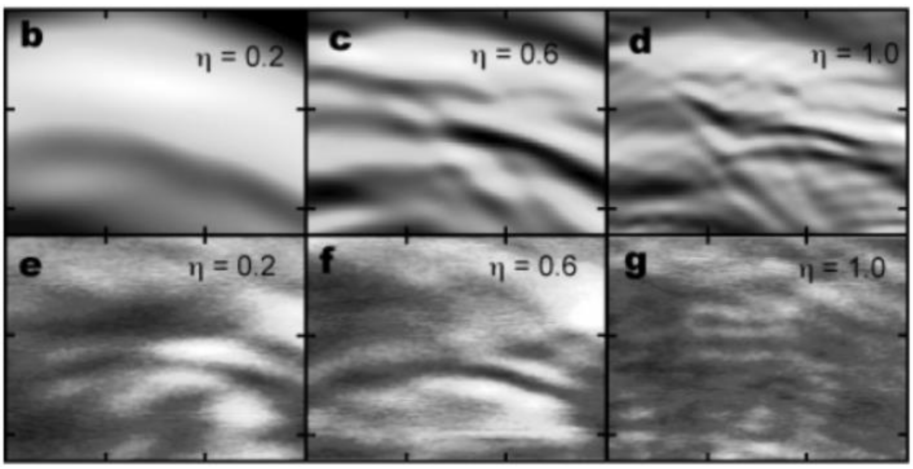


Figure 3. Interference fringes obtained via a modern experiment (Aidala et al., 2007).

2.3.2. MC2. Cutting a cone properly

One cannot exclude that a group of experts of the era have conceived a method of generating bronze or wooden conic surfaces, which subsequently cut (intersected) with a rigid plane. The associated action(s) are depicted in Fig. 4.

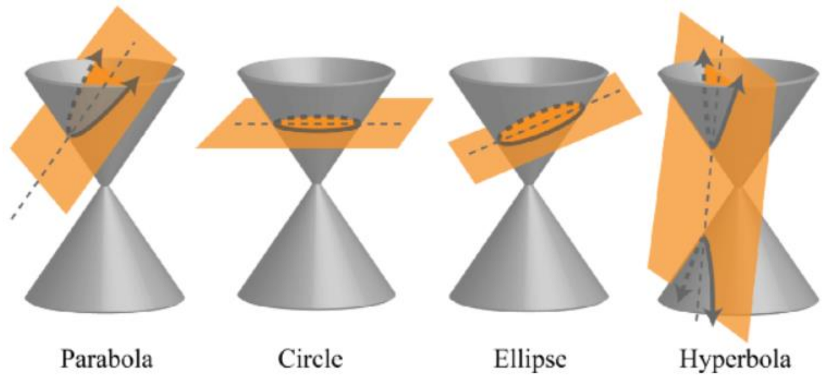


Figure 4. The procedure of generating all conics by cutting a right cone with a (rigid) plane (u/carattinim, 2017). The curve of main interest here, namely a hyperbola, results if the cutting plane is parallel to the axis of the cone.

2.3.3. MC3. The celebrated method of hyperbola construction proposed by Menaechmus

For completeness, we must include the celebrated method of hyperbola construction discovered and proposed by Menaechmus (see, for example, ("Duplication of the Cube," n.d.)), although this method had been proposed more than one thousand three hundred (1300) years after the first trace of this conic in the prehistoric Aegean. However, we firmly believe that this method of construction requires an axiomatic, theoretical approach of geometry, which includes notions and theorems exceptionally advanced for a prehistoric era. Hence, we strongly feel that this method of hyperbola construction could not have been developed in the Late Bronze Age.

2.3.4. MC4. Exploitation of the homocentric circles

Another strict definition of a hyperbola is that this curve is the locus of the points on a plane that have a constant absolute distance from two fixed points of the same plane. Equivalently, if E_1 and E_2 , called focuses, are two fixed points on a plane and M an arbitrary point on the same plane, then M belongs to a specific hyperbola if and only if the following relation holds:

$$|ME_1 - ME_2| = d \tag{3}$$

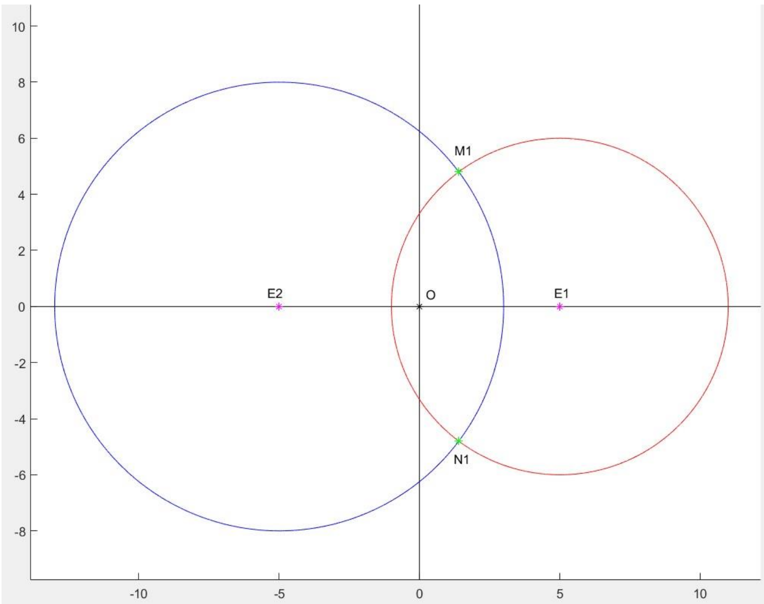
where difference $d > 0$, characterizes the hyperbola in hand.

One may use relation (3) above, in order to construct a hyperbola as follows:

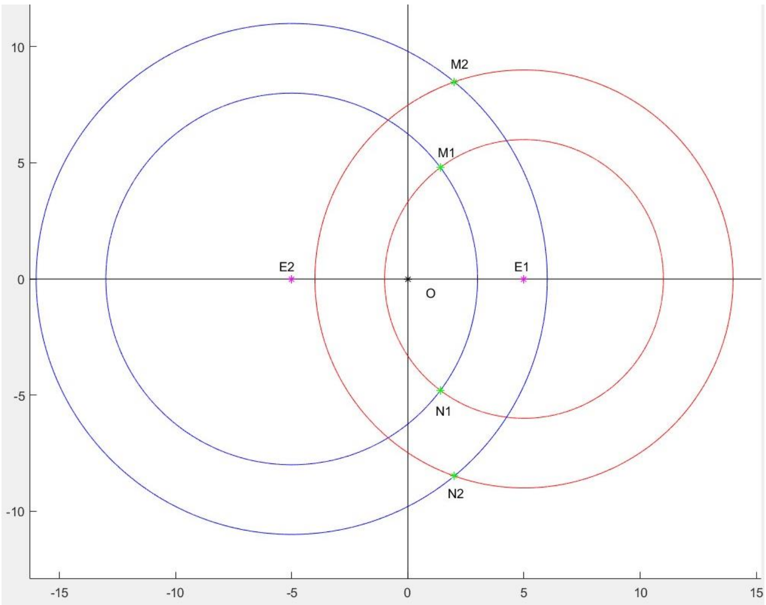
Step1. We choose two numbers R_2 and R_1, such that $R_1 > R_2, R_1 + R_2 \geq |E_1\ E_2|$.

Step2. We draw two circles, the first one with center E_1 and radius R_1 and the second one with center E_2 and radius R_2. If we let $d = R_1 - R_2$, then evidently, the two points of intersection M_1 and N_1, belong to the hyperbola defined via (3) (see Fig. 5 (a)).

Step3. We slightly increase both radii R_1 and R_2 by a particularly small positive quantity δR. We once more draw two additional circles, the first one with center E_1 and radius $R_1 + \delta R$ and the second one with center E_2 and radius $R_2 + \delta R$. The two new points of intersection M_2 and N_2, belong to the same hyperbola with points M_1 and N_1, which is defined via (3) (see Fig. 5 (b)).



(a)



(b)

Figure 5. (a) Two points M_1 and N_1 belonging to the hyperbola (3); (b) another pair of points M_2, N_2 belonging to the same hyperbola.

Step4. We continue in this way by adding more and more positive quantities δR in the radii of two circles always centered at E_1 and E_2.

In this way one may obtain as many points of the same hyperbola (3), as one desires (see Fig. 6). The smaller quantity δR is, the closer to a smooth hyperbola the obtained points M_i and N_i are.

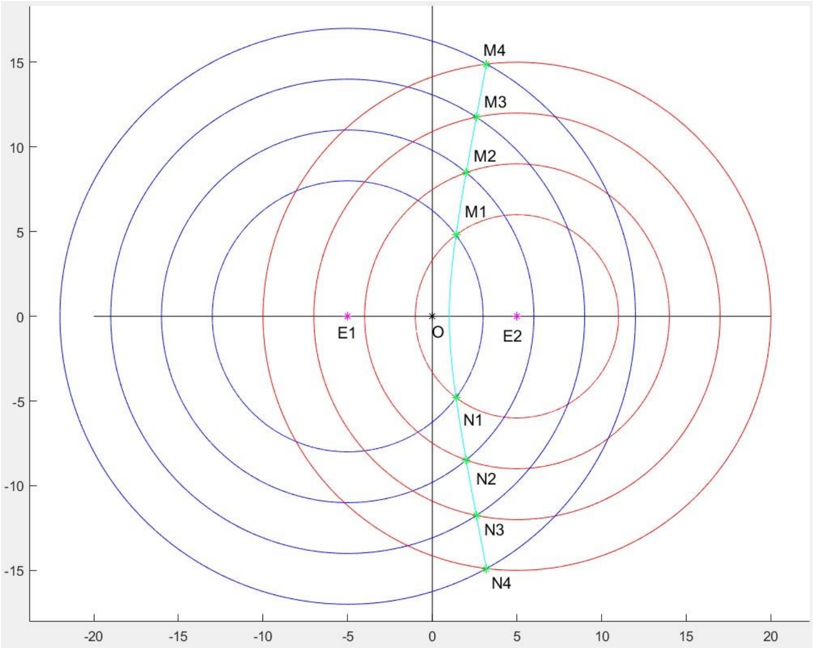


Figure 6. Additional points, generated as described in MC4, together with the hyperbola (shown in cyan), to which these points belong.

We tend to believe that the aforementioned process is a serious candidate or a basis for the actual method of construction of hyperbolae in the Late Bronze Age Aegean civilizations. We have this opinion for three main reasons:

a) we feel that this method of construction is not prohibitive for the era, although it requires a non-trivial amount of novelty.
b) To the best of our knowledge, configurations including homocentric circles have been found at Akrotiri, Thera.
c) Our research team has proved in (Papaodysseus et al., 2006b) that the prehistoric inhabitants of Akrotiri, Thera settlement, knew how to construct central angles of sequences of regular polygons, such as regular 8-gon, 16-gon, 32-gon but also of regular 6-gon, 12-gon, 24-gon, 48-gon, etc. The intersection of the corresponding straight lines bundles with a set of homocentric circles, generate points belonging to linear spirals (see Fig. 7).

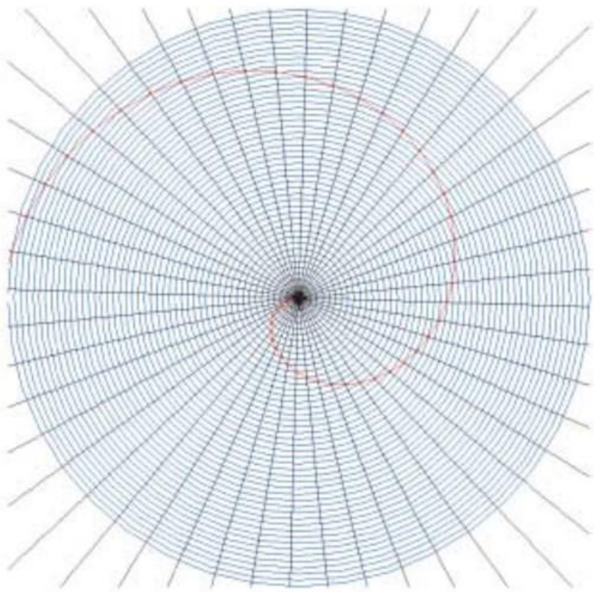


Figure 7. The points of intersection of a set of radii of regular polygons with homocentric circles, belong to a linear spiral.

2.3.5. Use of mechanical devices

One cannot exclude that technology experts of the era had exploited the method of construction based on homocentric circles so as to develop apparatuses including gears, capable of producing hyperbolae and linear spirals with a particularly high precision and smoothness. Such a method compatible with the technological level of the era, will be presented by the authors in a future publication.

3. THE HYPOTHESES CONCERNING THE WAY THE ARTIST(S) HAD DRAWN THE WALL-PAINTINGS AND A BRIEF DESCRIPTION OF THE METHOD DEVELOPED HERE FOR TESTING THEM

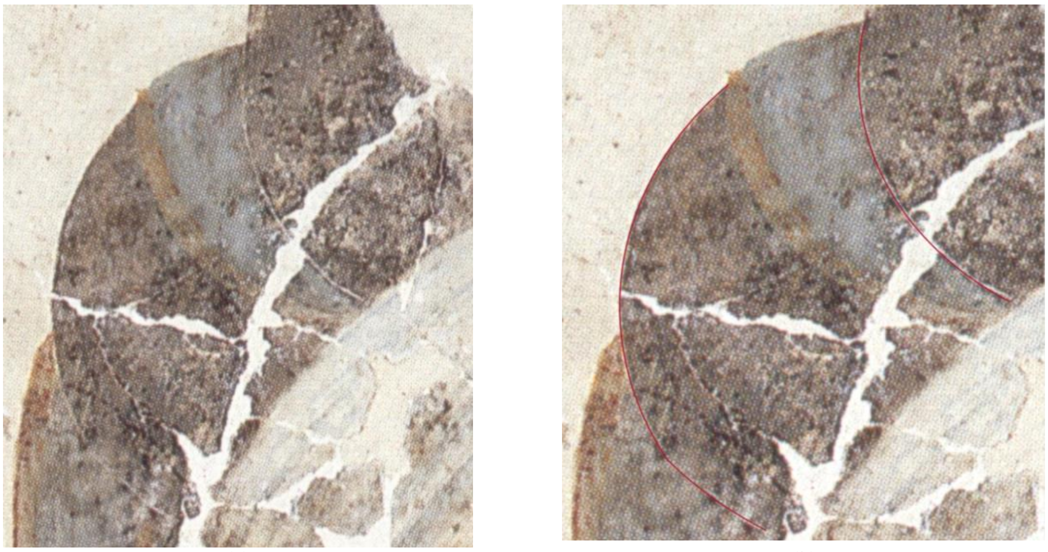
In the present Section, the authors will express their opinion concerning the manner with which the prehistoric artists were drawing the studied wall-paintings. The authors will also give a brief description of the methodology they have developed and applied for testing and verifying the aforementioned conjectures.

3.1. The Presumed Method of Drawing of the Considered Wall-Paintings in the Aegean During the Late Bronze Age

The authors have, so far, studied more than twenty-five (25) Late Bronze Age wall-paintings, quite meticulously, in order to determine the method of their drawing and, thus, have reached the conclusions presented below:

C1. One can, by no means, exclude the eventuality that the Late Bronze Age artists were using a small-scale template, as a pole star for the drawing of the actual synthesis on the wall. Furthermore, if such a template indeed existed and was used, it is quite likely that it was divided in sub-frames/sub-regions, so that the artists could follow the correct proportions in the substantially larger final synthesis on the wall.

C2. Next, it seems quite probable that the artist was selecting a guide that best satisfied his aesthetic criterion, he was fixing it on the wall and was engraving a first, rough draft of a corresponding border part. In fact, in various wall-paintings the authors have detected (sometimes by touch) various curves engraved on the plaster, which also correspond to parts of the six detected geometric guides, in a very good manner (see Fig. 8 (a), (b) and 9).



(a) (b)

Figure 8. (a) Two curves clearly engraved on the plaster; (b) the guides that generated the engravements shown in Fig. 8 (a) (photo scanned from book (Doumas, 1992)).

Figure 9. Another ensemble of engraved curves that have been generated by means of the spiral "ls2" introduced in Section 2 (photo obtained by the authors).

Subsequently, most probably, the artist was choosing a second geometric guide and, by means of it, continued engraving the designed border line on the plaster and so on.

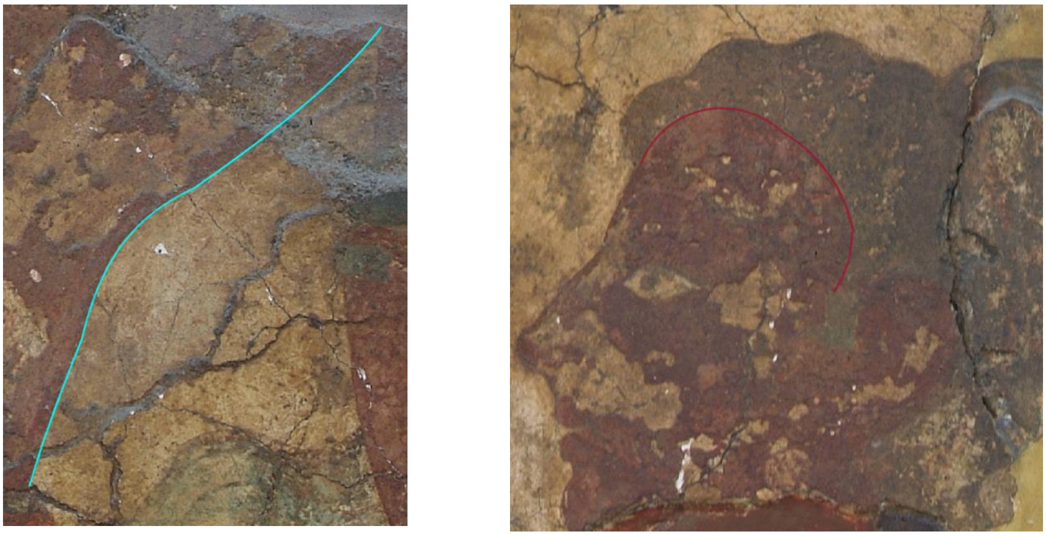
C3. As a next step, it seems that the artist covered every engraved, slightly rough, contour line on the plaster, with a coloured brushstroke. Once more, the brush was very likely guided by the corresponding/proper geometric stencil. In this way, the artist succeeded in generating a coloured, very stable and clear-cut contour.

We must point out that we have not spotted engravings in all studied wall-paintings (or at least we have not been able to do so); this gives rise to a considerable probability that numerous contour lines have been directly generated via a guided brush.

In fact, it seems very possible that the artists had at their immediate disposal and access a set of stencils and/or apparatuses; in order to draw a figure, they picked one of these guides and used it as a stencil, so as to draw a stable, clear-cut contour line of the figure on the wall. In other words, in this way a beautiful and "solid" brushstroke appeared on the wall. The authors believe that the guide or the apparatus that assisted the artists in drawing such stable and clear brushstrokes, was placed on the wall; actually, it was probably pinned up on it momentarily. This hypothesis seems to be fully supported by the fact that, in various wall-paintings, the authors have spotted numerous pretty small holes on the plaster, which were very plausibly generated when the artist pinned the guide on the plaster.

C4. As it has already been pointed out, after the generation of a brushstroke, the artist was proceeding to the next one by choosing another guide or another part of the same guide. Concerning the relation between two arbitrary successive brushstrokes, the authors strongly believe that the artists were trying to ensure the following:

i) For aesthetic reasons, the artist(s) was very meticulously trying to guarantee the minimum possible (spatial) distance of the end point of the first brushstroke and of the first point of the second one (see Fig. 10).
ii) With an incredible subconscious knowledge concerning the smoothness of the border line, the artist(s) was, again, meticulously trying to ensure what in contemporary Mathematics one calls "continuity of the tangent vectors at the beginning of the second brushstroke and the end point of the first one". For this action of the artist(s), we shall employ the expression "the artist(s) tried to guarantee the minimum angular distance between any two successive brushstrokes". Freely speaking, in contemporary Mathematics again, "the artist(s) subconsciously tried to ensure the maximum possible smoothness in the transition from the one stencil part to the adjacent one" (see Fig. 10).
iii) Wherever the artist(s) wanted to change the curvature of the contour line, then the artist(s) simply turned over the guide. Evidently, after performing such an action, the artist(s) was meticulously trying to respect the continuity conditions described in i) and ii) above. We believe and hope that all figures presented here, will help clarify the aforementioned actions of the artist(s) (see Fig. 10 below).



(a) (b)

Figure 10. (a) a first example demonstrating the particularly smooth transition from one guide part to another; the three (3) stencil parts correspond to the same guide and more specifically, in the hyperbola we have symbolized with the cyan color; (b) A second analogous example of smooth transition corresponding to three parts of the spiral "ls2".

Finally, the authors, despite the considerable number of wall-paintings' figures they have analytically studied (more than 25), they are, each time, stunned by the amount of artistry hidden behind each such drawing; they are also amazed by the great variety of themes that can be generated by the six stencils and the procedure introduced here.

3.2. *First Stage Processing of the Frescoes' Images and Definition of the "minimum error" and "maximum error"*

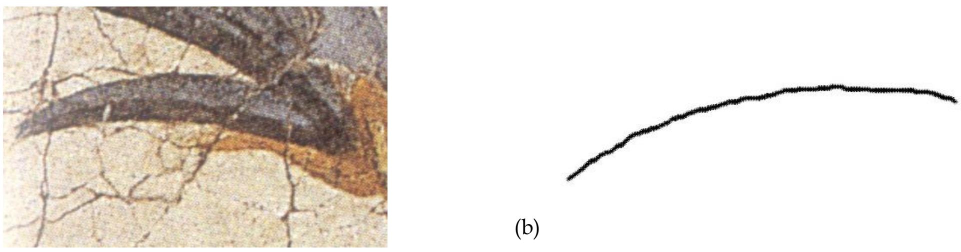
At first, the authors have obtained very good quality images of the studied wall-paintings, with a resolution of at least 27.5 pixels per centimetre; evidently, whenever the authors could not photograph the wall-paintings by themselves, the analysis of the image depended on its availability. Thus, with the exception of the "Prince of Lilies", the resolution of the other frescoes' images lies between 57 and 240 pixels per cm (px/cm).

Then, in connection with these images, the following steps have been applied:

i) the images have been segmented by means of the automatic method presented in (Papaodysseus et al., 2004), which however allows for user interaction.
ii) We have extracted the contours of all segmented figures appearing in each wall-painting. The contours have been extracted by a novel method that has been applied for the first time in connection with the considered large new set of wall-paintings.

In the subsequent Sections 4 and 5, we shall demonstrate that the entire set of contours of all studied wall-paintings impressively match the stencils introduced in Section 2.1, in a piecewise manner. Thus, we shall manifest that for each proper contour part CP of anyone of the treated frescoes, there is a corresponding, suitably selected, part of a certain stencil that optimally fits CP. In the present subsection, we shall clarify the meaning of "optimal matching" and of other, associated notions, like the "minimum error".

A typical result of actions A) and B) described above, is given in Fig. 11.



(a) (b)

Figure 11. (a) the upper left leaf of the first "Lillie", extracted from image S.M-1 of the Supplementary Material; (b) the upper contour of the leaf depicted in (a) above. The contour is shown in black asterisks and for each such part of the border line of a figure in a fresco, we shall employ the term "an actually drawn contour part".

Subsequently, we have tested if any part of the geometric prototypes introduced in Section 2.1, optimally fits the upper contour, say CP, shown in Fig. 11 (b). The corresponding optimal match has been achieved via minimization of "the approximation error", defined as follows:

Consider any geometric prototype part, say GP, which is subject to rotation and parallel translation, giving rise to a new position of it, say GP^{RT}, where superscript RT stands for Rotated-Translated. Also consider any point $(x_i, y_i) \in CP$; then, we compute the distance of point (x_i, y_i) from GP^{RT}, for all $(x_i, y_i) \in CP$, as shown in Fig. 12.

Figure 12. Computation of the distance of the arbitrary point $(\boldsymbol{x_i}, \boldsymbol{y_i}) \in \boldsymbol{CP}$ from $\boldsymbol{GP^{RT}}$: from this point, we draw the line vertical to curve $\boldsymbol{GP^{RT}}$, with point $\boldsymbol{\Pi(P, Q)}$ being their intersection. Then, the associated distance is $\boldsymbol{d_{i,GP^{RT}}} = \sqrt{(\boldsymbol{x_i} - \boldsymbol{P})^2 + (\boldsymbol{y_i} - \boldsymbol{Q})^2}$.

The overall distance of CP from an arbitrary position of GP^{RT} evidently is

$$d(CP, GP^{RT}) = \frac{1}{N}\sum_{i=1}^{N} d_{i,GP^{RT}} \quad (4)$$

where N is the number of pixels of CP; we shall call this overall distance as "the mean error" between CP and GP^{RT}. The optimal position of GP^{RT} in respect with CP is obtained as described in subsection 3.3 below. The corresponding process will also offer the proper rotation and parallel translation that must be applied to GP, so as its corresponding part optimally fits CP (see Fig. 13). When GP^{RT} is in the optimal fitting position, with respect to CP, then we shall use for it the symbol GP^{opt} (see Fig. 13). For the corresponding mean error $d(CP, GP^{opt})$ we shall employ the term "minimum (mean) error", while for the corresponding maximum error $(\max_i(d_{i,GP^{opt}}))$, where $i = 1, \ldots, N$ the number of pixels of CP, we shall use the term "maximum error".

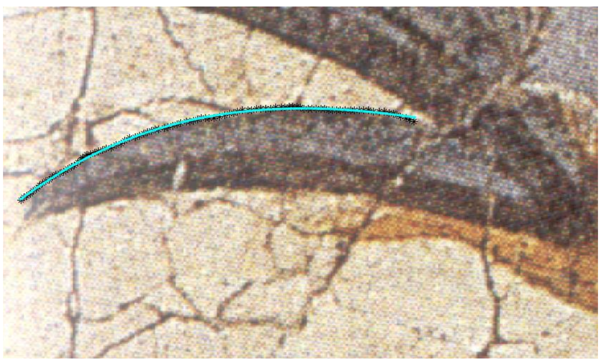


Figure 13. Optimal matching of the proper part of the "cyan" hyperbola to the actually drawn upper contour of leaf $\boldsymbol{CP}$. The corresponding overall distance, i.e., the associated mean error is 0.13 mm, while the corresponding maximum error $(\max_i(\boldsymbol{d_{i,GP^{opt}}}))$ is 0.52 mm.

As it will become evident from all associated Tables included both in the main text and in the Supplementary Material, it always holds that $d(CP, GP^{opt}) \leq 0.3$ mm and $\max_i(d_{i,GP^{opt}}) < 0.8$ mm; actually, in the overwhelming majority of optimal fits, these inequalities become more strict, namely $d(CP, GP^{opt}) \leq 0.25\ mm$ and $\max_i(d_{i,GP^{opt}}) < 0.7$ mm. The same inequalities hold true in connection with all wall-paintings we have studied so far.

3.3. *A brief Description of the Methodology Developed by the Authors for Testing All Associated Conjectures*

As a next step, we have taken the following actions:

i) On the contour CF of each figure, isolated contour-parts (CP) have been determined by defining as endpoints of CP, those points of CF where a serious breach of continuity has been automatically spotted.

ii) The developed system got through each such contour part and determined a corresponding part of a specific geometric prototype that best fitted to it. The associated optimal fitting obeyed the following criteria:

Criterion 1. The "mean error", namely the average distance $d(CP, GP^{opt})$, should always remain smaller than 0.3 mm (see all figures of Sections 4 and 5).

Criterion 2. The maximum discrepancy-distance of the actually drawn contour from the corresponding geometric prototype should always remain smaller than 0.8 mm. We would like to emphasize that these upper bounds for the "mean error" and the maximum one, have been chosen for the following reasons:

a) All wall-paintings tested so far, satisfied these conditions; therefore, we have extended the associated demand on every new wall-painting we have studied and
b) the overall aesthetic result of the resulting wall-paintings, is indeed impressively good as far as these upper bounds were satisfied. On the contrary, whenever one of these two thresholds was violated the resulting fresco's contour was clearly sub-optimal. One may consult Figs 15 and 22, as well as S.M-3, S.M-6, S.M-11, S.M-12, S.M-13, S.M-19 and S.M-22 where the determined stencils' parts GP^{opt} are depicted standing alone; from these Figures, one may immediately deduce that the approximation of the actually drawn contours by the optimally fit to them GP^{opt}, is indeed excellent.
c) We shall demonstrate in Sections 4 and 5 that a quite large subset of optimally fit GP^{opt} to the actually drawn contours are unique from a mathematical point of view. For all these unique parts of the geometric prototypes the introduced upper bounds for the minimum (mean) and maximum error always hold; actually, in most cases, the corresponding distances $d(CP, GP^{opt})$ are substantially smaller (e. g. see Fig. 13 and all related Tables in the main text and the Supplementary Material).

Criterion 3. Each contour part should be covered by the minimum possible number of prototype parts. Equivalently, the system asked for the maximum possible length of each stencil's appearance.

Criterion 4. In the case where more than one stencil parts covered an actually drawn contour-part CP, continuity between the stencils' parts should be guaranteed with the same norms that have been stated in sub-section 3.1, C4 (see Fig. 10). In fact, we ask for the minimum possible angular distance between the unit tangent vector at the endpoint of the $\nu - th$ of GP^{opt} and the first point of the $(\nu + 1) - th$ of the next GP^{opt}.

iii) The prototype parts have been fitted to the corresponding actually drawn contour-parts via optimal rotation and parallel translation. This action has been achieved by a pair of methods which constitute an improvement of the methods introduced in (Papaodysseus et al., 2008, 2004) and in particular via a combined Lagrange minimization procedure, which considers all aforementioned demands. It is interesting to stress that the aforementioned rotation and parallel translation, in essence imitates the actions-movements the prehistoric artist(s) was, most probably, making in order to draw the fresco, as these actions have been described in sub-section 3.1.

4. ESTABLISHING THE METHOD OF DRAWING OF A SET OF FOUR (4) FRESCOES, EXCAVATED AT AKROTIRI THERA BELONGING TO THE LATE BRONZE AGE

In the present Section the authors will substantially support that the four (4) frescoes unearthed at Akrotiri, Thera, namely "Lilies", "The griffin", the "Middle figure of the Adorants" and "The Fisherman" more analytically presented in subsections 4.1 to 4.4 below and in Section S.M.2 of the Supplementary Material, had been drawn by means of the guides/stencils and the method introduced in Sections 2 and 3 ((Doumas, 1992; Sakellarakēs and Ntumas, 1994; *The Oxford Handbook of the Bronze Age Aegean*, 2012)).

4.1. Study of the method of drawing of the fresco "Sea Daffodils" or "Lilies"

The "House of the Ladies" is one of the northernmost buildings to have been uncovered at the site of Akrotiri (Doumas, 1992). In "Room 1" of this edifice, a wall-painting is extended over three walls: the south one, the west wall and the north one. An image of this wall-painting, together with a brief description of it, based on (Baumann et al., 1993; Doumas, 1992) , is given in S.M.2.1 of the Supplementary Material.

In Figures 14, 15, 16, below it is demonstrated that the boundaries of all objects appearing in the wall-painting "Sea Daffodils", fit the very same stencils introduced in Section 2, with exceptionally low minimum (mean) and maximum error. In addition, the very same figures satisfy all the prerequisites introduced in Section 3, associated with the hypothesized method of drawing of all studied wall-paintings.

In particular, all the stencils' parts shown in Fig. 14 fit the actually drawn contour in an excellent manner, namely with an impressively small minimum (mean) error, as well as a pretty small maximum one. The corresponding fitting errors are shown in Table 4.

Moreover, in Fig. 15, the entire set of the parts of the geometric guides employed for the drawing of all objects appearing in "Sea Daffodils", are presented standing alone, so as to form the entire contour of all involved figures. From this figure it is evident that the determined stencils' parts alone, fully and reliably represent the borders of the ensemble of "Lilies". Finally, the stencils' parts that are "unique" from the mathematical point of view are shown in Fig. 16; we shall clarify the meaning of the latter term immediately below.

Indeed, we use the term "unique", in order to describe the fact that no other "acceptable" Geometric-Mathematical curve match the corresponding brushstroke contours. We have previously employed the adjective "acceptable", in order to make clear the following: in the quest for the possible stencils that had been used for the drawing of the studied wall-paintings, we have constrained ourselves in geometric curves that they were known and they had been studied in the Classical and Hellenistic periods. Consequently, we have confined our analyses to:

a) The curves that, in contemporary Mathematics one may say that correspond to polynomial equations up to the second degree; more analytically, such curves are the straight-line segments, the circle, the ellipse, the parabola, the hyperbola (i.e., the conics).
b) We have also "dared" to consider more advanced curves, such as the cissoid discovered by Diocles (e.g. see (Lawrence, 2014)), the cycloid etc. The curves that have offered the optimal results, are precisely those presented in Section 2.
c) The spirals that had been also studied and analyzed, are those treated in the Classical and Hellenistic Ages, such as the exponential spiral, the linear one, the spiral that emerges when one unwraps a rope initially wrapped around a peg and few others.

We would like to make clear that, again, the spiral that by far has offered the optimal results, is the Archimedes-linear spiral and in particular its two versions introduced in Section 2.

We would also like to stress that uniqueness is strongly associated with the prerequisite-demand of the existence of a minimum number of guides that, practically, might have generated the entire ensemble of contours of all studied wall-paintings. In fact, one may divide the border of each figure in a different manner and for each such contour part, one could perhaps find a different stencil to fit. However, in this way, many tenths or even hundreds of different guides could emerge for each wall-painting separately. Evidently, for the numerous frescoes that the authors have studied, this would lead to the conclusion that many hundreds or even thousands of guides had been used for the drawing of all these magnificent syntheses; clearly, such an approach is completely unacceptable.

In addition, we emphasize that the curves matching the Lilies' contours in Figures 14, 15 and 16 indisputably reinforce the correctness of the hypothesis that the spotted stencils had indeed been used for this drawing. This last statement is fully and undoubtedly confirmed by the fact that, once more, the average of the minimum errors between the geometric guides and the associated actually drawn contours of the entire fresco, is less than or equal to 0.26 mm. In an analogous manner, the related maximum errors are smaller than or equal to 0.65 mm. The corresponding errors are shown in Table 4 both for the entire set of optimally fit stencils' parts and for the unique ones.

Finally, concerning the exact type of guides, we might have chosen far more complicated contemporary curves, such as those that correspond to the higher order degree polynomials, complicated trigonometric functions etc. However, these curves slightly lower the distance $d(CP, GP^{opt})$ between the stroke and the prototype curve; if one also considers the enormous amount of novelty and complexity required for the construction of those "contemporary" stencils, then one may exclude them immediately and safely.

All these figures and all previous remarks imply that the covering of the contours of the drawn figures by means of the parts of the geometric guides shown in Fig. 14, is extremely improbable to be accidental. On the contrary, it indisputably upholds our conjecture that the wall-painting "Sea Daffodils", had been drawn via the use of the aforementioned geometric stencils.

***Figure 14.** The contours of the objects appearing in the wall-painting "Sea Daffodils" (depicted in image S.M-1 of the Supplementary Material), with all corresponding, optimally determined guides' parts on them. We, once more, emphasize that each colour uniquely stands for one and only one geometric prototype.*

***Figure 15.** The parts of the geometric guides of Figure 14 standing alone. From this Figure, one may immediately conclude that these geometric sub-curves practically fully and reliably constitute the well-preserved and conserved border parts of the corresponding Figure S.M-1 and 14.*

Figure 16. The subset of the geometric guides' parts shown in Figure 14 and 15, which are "unique" according to the previous analysis introduced in the current subsection.

Table 4. Numerical characteristics of the determined geometric guides' parts, such as the number of occurrences, the minimum and maximum length and the mean value of the minimum error and the maximum error, for each type of guide separately a) concerning all guides' parts, b) regarding only the unique ones. As defined in Section 3, the term "minimum error" is used in order to describe the minimum (mean) distance $d(CP, GP^{opt})$; evidently the term "maximum error" refers to $\max_i(d_{i,GP^{opt}})$, where $i = 1, \ldots N$ the number of pixels of CP.

Type of Stencil	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	30	5	5.3	8.09	21.06	21.06	0.15	0.21	0.43	0.59
c	69	20	1.95	4.84	19.55	19.55	0.15	0.16	0.44	0.48
g	40	9	3.2	6.33	23.12	13.71	0.15	0.2	0.41	0.52
m	5	3	12.91	12.91	20.92	20.92	0.23	0.26	0.58	0.65
ls0	0	0	-	-	-	-	-	-	-	-
ls2	285	0	0.19	-	5.73	-	0.08	-	0.23	-

4.2. *Study of the Method of Drawing of the Figure "The griffin" Belonging to the Synthesis "The Crocus gathering"*

Following C. Doumas (Doumas, 1992) meticulously, an excellent assemblage was decorating the east and the north wall, in the storey directly above the "Lustral Basin". This assemblage has been restored and the content of the synthesis is now clear; this fresco is the celebrated "Crocus Gathering" ("Κροκοσυλλέκτριες"). The mythical creature of griffin (γρύπας) belongs to this synthesis and it is briefly described in S.M-2 in the Supplementary Material.

In the present work, we shall demonstrate that the border lines of the figure griffin, shown in Fig. S.M-2, are exceptionally well "covered" by the spotted geometric guides, in a piecewise manner.

Below, in Fig. 17, the figure of this mythical creature is shown with all corresponding guides' parts optimally matched and depicted on its contour. We repeat that in Fig. 17, each different colour uniquely corresponds to a specific guide and that all conditions analytically introduced in Section 3 have been meticulously applied in the procedure of determining the proper stencil parts.

Figure 17. The griffin's contour with all the geometric guides' parts optimally fit on it. The contour covering satisfies all the criteria introduced in Section 3.

Moreover, in Fig. S.M.-3 of the Supplementary Material, the actual griffin's figure contour is shown standing alone, exclusively formed by the determined geometric guides' parts (GP^{opt}). Finally, in Fig. S.M-4 of subsection S.M.2.2 of the Supplementary Material, the parts of the guides that are unique in accordance with the analysis introduced in subsection 4.1, are depicted.

Conclusively, we must emphasize that all remaining figures belonging to the synthesis "Gathering of Crocus", have, most probably, also been drawn by means of the very same guides. The authors have already demonstrated in (Papaodysseus et al., 2022, 2006a), that this conjecture holds true in connection with three (3) female figures, including the "Goddess". We will support the hypothesis that the remaining objects appearing in the specific wall-painting had been also drawn via these stencils, in a future publication.

4.3. *How the Middle Figure of the Synthesis "Adorants" in the Lustral Basin of Xeste 3 Had Been Drawn*

In this subsection, we shall deal with the most probable method of drawing of a figure of another important synthesis, nowadays called "The Adorants", initially decorating room 3a, located at the ground floor of Xeste 3. A brief report associated with this synthesis is given in the Section S.M.2.3 of the Supplementary Material, which is entirely based on (Doumas, 1992; Sakellarakēs and Ntumas, 1994). The middle female figure of "The Adorants", is depicted in Fig. S.M-5 of the Supplementary Material. It concerns a female, depicted entirely in profile, sitting on a small knoll, and slightly bent over.

In Fig. 18, the aforementioned woman is shown with the entire set of contours of her figure, covered by the determined, proper geometric guides' parts. We, as always, emphasize that in Fig. 18 each different colour uniquely corresponds to a specific geometric prototype and that all conditions analytically introduced in Section 3 and sub-section 4.1 have been meticulously applied.

Figure 18. A female figure belonging to the synthesis "The Adorants", which was decorating the "Lustral Basin" in Xeste 3 walls, with all corresponding, optimally determined guides' parts on it.

Once again, in Fig. S.M-6 of the Supplementary Material, the actually fit parts of the geometric guides of the specific female drawing is shown standing alone, while in Fig. S.M-7 of the Supplementary Material the parts of the guides are shown that are unique, according to the analysis introduced in subsection 4.1.

We stress that the geometric curves matched the border lines of this female figure, constitute a decisive proof that this figure, too, has, most likely, been drawn via the use of the stencils/guides introduced in Section 2 and by means of a method pretty analogous to the one described in Section 3. The previous assertions are completely and indisputably confirmed by the fact that, once more, the minimum (mean) error $d(CP, GP^{opt})$ between the geometric guide and the drawn contour of this female figure is less than 0.3 mm, while the corresponding maximum error (discrepancy) is less than 0.75 mm (see Table S.M-2 in the Supplementary Material). In addition, Figs 18 and S.M-6 clearly indicate the eventual actions that had been taken by the artist(s), in order to draw the specific Adorant.

4.4. The Corresponding Analysis Concerning the Wall-Painting Named "Fisherman"

Of all buildings unearthed at Akrotiri, the "West House" is the one which, so far, has been most thoroughly investigated ((Doumas, 1992; Sakellarakēs and Ntumas, 1994)). In "Room 5" of the West House, there was a figure, nowadays named "Fisherman", located on the north wall. This wall-painting was discovered in a rather good condition.

In the present subsection, we shall manifest the most probable method of drawing of this fresco and we shall support that it is exactly the same with the one applied to all studied wall-paintings, so far. In order to better support this statement, we have obtained three sub-images of the "Fisherman" with a substantially higher resolution. The entire image of this figure and of the aforementioned three (3) sub-images are given in Figures S.M-8, S.M-9 and S.M-10 of the Supplementary Material, respectively. In the corresponding subsection S.M.2.4, an outline of this wall-painting is given, which is meticulously based on book (Doumas, 1992). As in the previous subsections 4.1, 4.2 and 4.3, Figures 19 and 20 (a), (b) and (c), below, manifest the excellent piecewise manner with which the proper parts GP^{opt} of the geometric prototypes presented in subsection 2.1, cover all the actually drawn borders of the "Fisherman".

We would like to emphasize that, once more, all these stencils' parts match the actually drawn contour in an excellent manner (see Table S.M-3 of subsection S.M.2.4). All these imply that the covering of the contours of the drawn objects in the fresco "Fisherman", by means of the parts of the geometric guides shown in Figs 19 and 20, it is very unlikely to be accidental. On the contrary, they strongly support the validity of our conjecture that this wall-painting had been drawn via the use of the aforementioned geometric stencils and with a method pretty similar to the one introduced in Section 3 and analysed in the previous subsections 4.1, 4.2, 4.3, too.

To further reinforce this statement, in Figs S.M-11, S.M-12 and S.M-13 we show the contours of the "Fisherman" and the fishes alone, formed entirely by the stencils' parts presented in Figs 19 and 20 (a), (b) and (c). Finally, in Fig. S.M-14 (a), (b), we present the guide parts of Figs 19 and 20 (b), which are "unique", as described previously.

Figure 19. The contour of the figure "Fisherman" depicted in image S.M-8 with all corresponding, optimally determined guides' parts on it.

(a) (b) (c)

Figure 20. (a) The border lines of the Fisherman's head depicted in image S.M-10, with all corresponding, optimally determined guides' parts on it; (b), (c) The contours of the fishes depicted in images S.M-9 (a) and (b), with all corresponding, optimally determined guides' parts on them.

5. DEMONSTRATION THAT MANY MINOAN, CRETAN FRESCOES HAD BEEN DRAWN BY THE SAME METHOD AND GUIDES AS THE PREVIOUS ONES OF AKROTIRI

In the present Section, we shall establish that a set of very well-known Minoan wall-paintings and/or corresponding fragments, all excavated in the island of Crete, had been most likely drawn with exactly the same stencils introduced in Section 2 and via methods quite similar to those hypothesized in Section 3. In other words, immediately below, it will be demonstrated that the contours of the frescoes "The Rhyton-Bearer", as well as the celebrated wall-paintings "Prince of Lilies", which is a name most probably referring to three different fragments of frescoes and "The blue bird", optimally match the detected geometric guides, in an excellent, piecewise manner.

5.1. *Study of the Wall-Painting "The Cup-bearer" or "The Rhyton-Bearer"*

One of the most famous Knossian wall-paintings and one of the best preserved, is the "Rhyton-bearer" ("ο Ρυτοφόρος") which belongs to a many-figures procession (see figure S. M-15 of the Supplementary Material). In this synthesis the drawn figures most probably are gift-bearers advancing in two opposite directions towards the central figure of a goddess or priestess. The wall-painting was most likely drawn during Late Minoan II, circa 1450 B. C., according to Evans, Cameron, Immerwahr and Hood (Cameron, 1976; Evans, n.d.; Hood, 2005; Immerwahr, 1990). However, we would like to emphasize that radiocarbon-based methods have estimated that Late Minoan II period was between 1550 and 1500 B. C. (Hood, 2005); we have no reasons at all, in general, to doubt the aforementioned results of the Exact Sciences.

In subsection S.M.3.1 of the Supplementary Material, we present an image of "the Rhyton-Bearer" (see Fig. S.M-15), together with a brief description of this image, based on the books (Sakellarakēs and Ntumas, 1994) and (Dimopoulou - Rethemiotaki, Nota, 2005).

In addition, in order to better reveal the method of drawing of this fresco, we have artificially divided the image shown in Fig. S.M-15 in three slightly overlapping sub-images, of a substantially greater resolution; these sub-images are also depicted in the Supplementary Material, in Figures S.M-16, (a), (b) and (c).

Next, in the three sub-Figures 21 (a), (b), (c) we once more demonstrate the excellent piecewise match of all actually drawn contours appearing in these higher analysis sub-images S.M-16, (a), (b) and (c) to the stencils presented in Section 2. From these figures and the associated Table 5, it is evident that the very same geometric guides with those of Akrotiri, optimally fit the actual drawn border lines of "The Rhyton-Bearer".

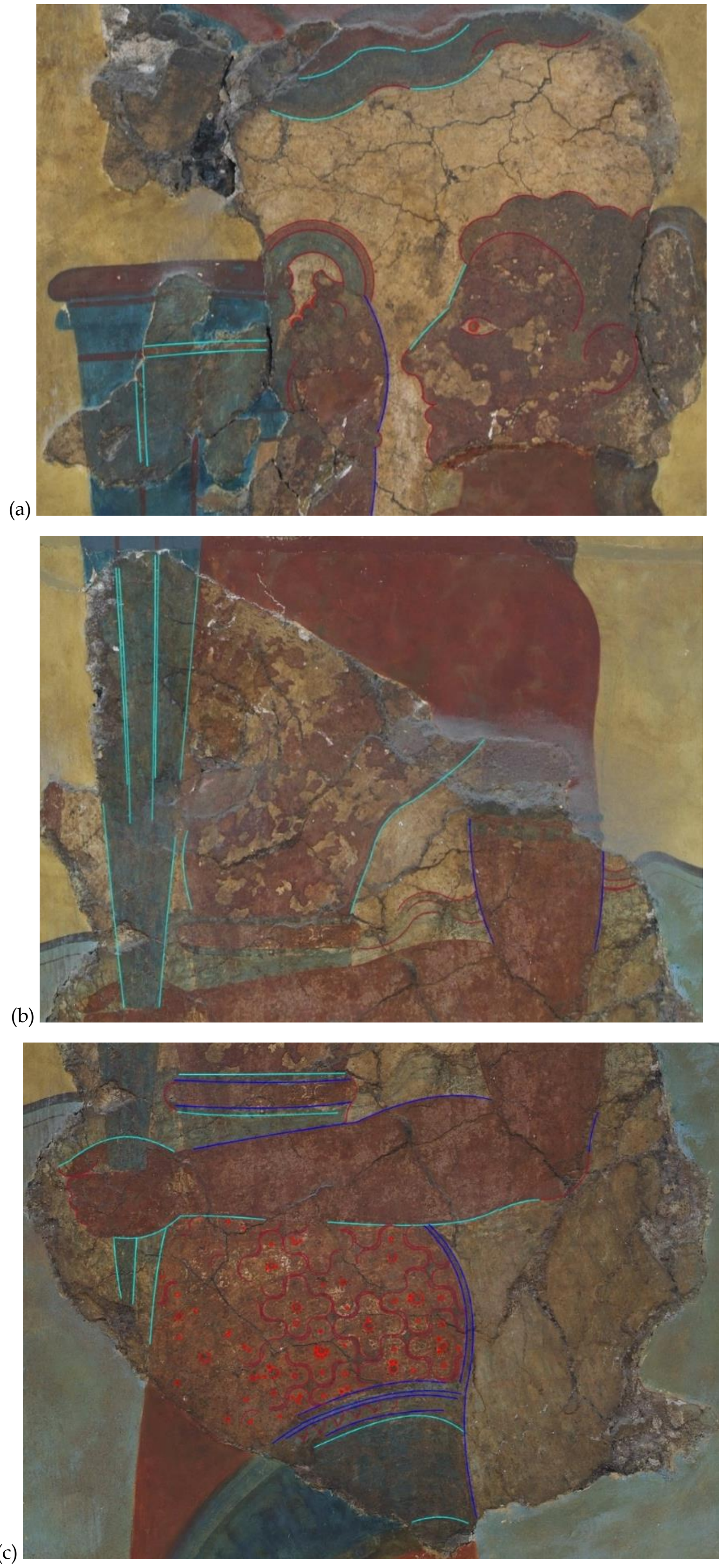
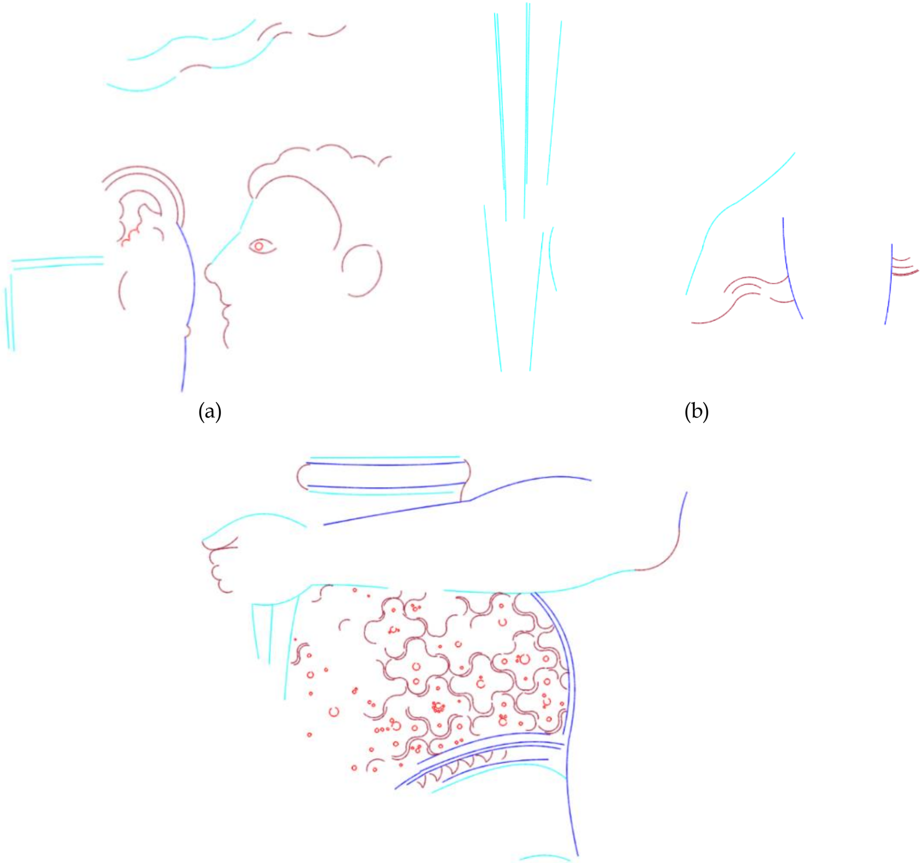


Figure 21. (a) The actual contours of the figure depicted in image S.M-16 (a), with all corresponding, optimally determined guides' parts on them; (b) The actually drawn contours appearing in Figure S.M-16 (b), covered by all corresponding, optimally determined guides' parts on them; (c) The actual border lines of the figure depicted in image S.M-16 (c), with all corresponding, optimally fit guides' parts on them.

Subsequently, in sub-Figures 22 (a), (b), and (c), we show the guides' parts appearing in Fig. 21, which form the border line of all figures of these drawings, standing alone. Finally, in sub-Figs 23 (a), (b) and (c), the subset of these stencils' parts that are unique are shown, in precisely the same sense as the one described in sub-section 4.1.



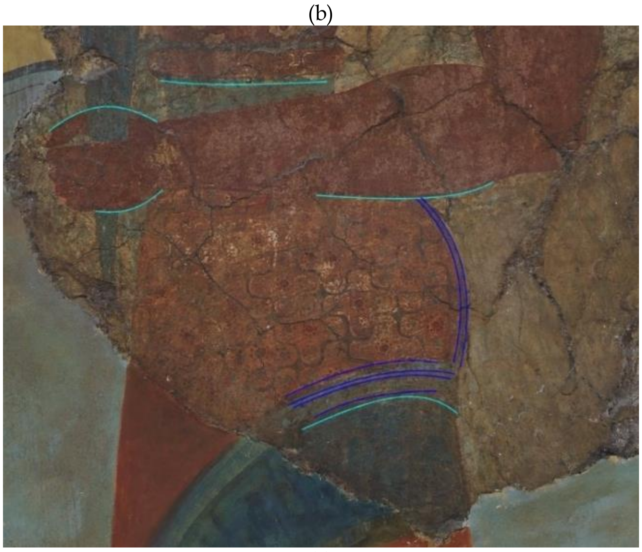
(a) (b)

(c)

Figure 22. (a) The parts of the geometric guides of Figure 21 (a), standing alone; (b) The parts of the geometric guides of Figure 21 (b), standing alone; (c) The parts of the geometric guides of Figure 21 (c) standing alone. One may equivalently state that it is obvious that these geometric sub-curves generate a "skeleton" of Figure 21 in an excellent manner. From the above Figures (a), (b) and (c) there is no doubt that these geometric sub-curves practically, completely and precisely represent the well-preserved and conserved contour parts viewed in sub-Figures S. M-16 (a), (b) and (c), respectively.

(a)

(b)



(c)

Figure 23. The sub-set of the geometric guides appearing in sub-Figures 21 (a), (b) and (c) which are unique, respectively.

The excellent way with which the detected stencils' parts GP^{opt} fit the corresponding, actually drawn contours CP, in all three sub-images, is manifested in Table 5 below.

Table 5. The already standard numerical characteristics concerning Figures 21, 22 (for all stencils' parts) and Figure 23 (for the unique ones only).

Type of Stencil	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	18	8	1.88	9.84	14.31	14.19	0.17	0.21	0.49	0.55
c	35	11	1.83	3.57	22.47	13.97	0.21	0.2	0.48	0.51
g	0	-	-	-	-	-	0	-	-	-
m	0	-	-	-	-	-	0	-	-	-
ls2	185	2	0.61	1.2	12.49	2.03	0.17	0.14	0.37	0.43
ls0	79	0	0.41	-	2.45	-	0.08	-	0.23	-

5.2. *Regarding the Wall-Painting(s) "the Prince of Lilies"*

According to the Greek Ministry of Culture ("Ministry of Culture and Sports | Heraklion Archaeological Museum," n.d.), this most important fresco from Knossos (which is presented in Fig. S.M-17 of the Supplementary Material) was found in fragments and it has been restored by Evans and his collaborators. The figure was named "Prince" because it was thought to represent the "Priest-King" who presumably lived in the so-called "Knossos Palace".

However, if one carefully considers the synthesis restored by Evans and his team, then one may observe that the only preserved, undivided sections are: (a) the crown and the upper part of the body (see Fig. S.M-18 (a)), (b) the middle part of the body that includes a part of the thigh (see Fig. S.M-18 (b)) and (c) the lower part of the body containing one leg and a part of the associated knee (see Fig. S.M-18 (c)). Equivalently, as we will more analytically discuss in subsection S.M.3.2 of the Supplementary Material, it seems very probable that each one of the three aforementioned fragments belong to different syntheses. For this reason, the authors of the manuscript in hand, show three different well-preserved "islands" of fragments in Fig. S.M-18, which they have called "top island", "middle island" and "lower island", respectively. We emphasize that in these "islands", we have not included the interventions made by Evans and his collaborators/restorators.

Figure 24. (a) The actual contours of the figure depicted in image S.M-18 (a), with all corresponding, optimally determined guides' parts on it; (b) The contours of the figure depicted in image S.M-18 (b), fully covered by the proper, optimally determined guides' parts on it; (c) The actual contours of the figure depicted in image S.M-18 (c), with all corresponding, optimally determined guides' parts on it.

Consequently, the basic conjectures introduced in the present manuscript, will not be applied to the entire synthesis restored by Evans and his collaborators, respecting the observations of Cameron, 1976; Niemeier, 1986; Shaw, 2004. On the contrary in Figures 24 (a), (b) and (c), we, once more, demonstrate the excellent piecewise match of all actually drawn contours of these Minoan Crete frescoes to the stencils presented in Section 2. As always, each colour uniquely corresponds to one (and only one) such geometric stencil. From these Figures and Tables S.M-4, S.M-5 and S.M-6, it is, once more, very likely that our hypothesis concerning the method of drawing of all studied wall-paintings, holds true for "The Prince of Lilies", too.

Subsequently, in Figures S.M-19 (a), (b) and (c) of the S.M., we show the guides' parts of figures appearing in 24 (a), (b) and (c) of the main text respectively that alone form the border lines of the figures of these frescoes. Finally, in Figures S.M-20 (a), (b) and (c) of S.M., the subset of these stencils' parts that are unique are shown.

5.3. *Establishing the Method of Drawing of the Minoan Fresco "The Blue Bird"*

This wall-painting is one of the earliest and most important artifacts of the Minoan Crete. It has been assigned to the Late Minoan IA period (LM IA), during the second half of the 16th century B. C., by (Hood, 2005; Warren and Hankey, 1989). Nevertheless, we must stress that radiocarbon-based methods (Friedrich et al., 2006) have estimated that the LM IA period was in the 17th century B. C. and, as always, the authors of the present work have no reason to doubt the results of the Exact Sciences.

A part of the specific assemblage including the "The Blue Bird" is shown in Fig. S.M-21 of the Supplementary Material, together with a brief description of it in subsection S.M.3.3, based on (Cameron, 1976; Hood, 2005; Sakellarakēs and Ntumas, 1994).

Again, the contours of all the well-preserved shapes appearing in Fig. S.M-21 that had been indeed drawn in the specific prehistoric era, are optimally approximated in a piecewise manner, by corresponding parts GP^{opt} of a stencil introduced in subsection 2.1 (see Fig. 25). Equivalently, the well-preserved and correctly conserved border lines of the figures appearing in this synthesis, are completely covered by a set of suitable parts of these six geometric curves.

Figure 25. The actual contours of the figure depicted in image S.M-21, with all corresponding, optimally determined guides' parts on them.

In subsection S.M.3.3 of the Supplementary Material, we also present an image including the detected guide parts GP^{opt}, standing alone (Fig. S.M-22), as well as the guide parts GP^{opt} appearing in Fig. 25, which are unique (Fig. S.M-23). Finally, we repeat that the matching of these parts of the mathematical curves to the actually drawn contour parts, is excellent (see Table S.M-7 and Figs S.M-22, and S.M-23).

6. CONCLUSIVE OBSERVATIONS, REMARKS AND DEDUCTIONS

6.1. *Very Likely Deductions Concerning the Method of Drawing of the Studied Frescoes*

A first, very important remark fully compatible with the previous analysis is the following: from the standpoint of Mathematics, Engineering and the associated Exact Sciences, it seems extremely probable that all the wall-paintings studied by the authors in the present work and in previous ones, had been drawn by the method described here. Equivalently, it is rather safe to assume that inhabitants of the Aegean Islands in the Late Bronze Age, had constructed the specific six (6) geometric prototypes, at least one thousand three hundred (1300) years, before the so far presumed conception and rigorous study of these curves in the Classical Age. Subsequently, according to our hypothesis, the Aegean prehistoric artists used these geometric prototypes, as guides for drawing all these marvellous wall-paintings. In doing so, they subconsciously, instinctively and emotionally followed a sequence of actions being in accordance with the Criteria introduced in Sections 3 and 4 above, in a "modern mathematical language". Of course, the substantial goal of those artists was to draw a synthesis, as fast as possible on the wet plaster, which satisfied the sense of aesthetic appearance and artistry of each one of them.

There are numerous reasons, which support our hypothesis that most of the wall-paintings unearthed at Akrotiri, Thera and Crete, belonging to the Late Bronze Age, had been drawn by the method introduced here. Indeed,

I. In a large number of frescoes (more than fifteen (15) frescoes of Akrotiri and more than twelve (12) frescoes excavated at Crete), the depicted figures' borders match parts of the six stencils presented in Section 2, in an excellent manner.
II. In most of the figures appearing in these frescoes, there are numerous contour parts, which are "unique" (see the related analysis in subsection 4.1). Loosely speaking, this term conveys the information that:
    a) other mathematical curves, far more modern, are prohibitively complex not only for the Late Bronze Age, but for the Classical Period, too. In addition,
    b) the improvement of the quality of matching between the prototype parts in one hand and the actual brushstroke on the other is practically negligible.
    c) The guides presented in Section 2, constitute the minimum possible number of prototype curves that fit the entire borders of all studied frescoes, in a such very good manner.

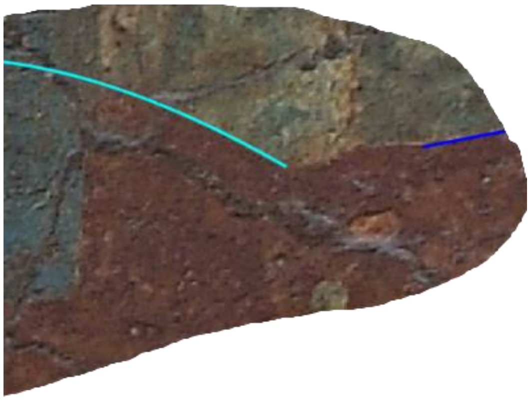
Consequently, in any case, the previous remarks strongly support the hypotheses that each such unique brushstroke optimally corresponds to a single stencil part and more specifically to the one that is, each time, presented here.

III. Wherever a detected part of a geometric guide did not fit the corresponding brushstroke contour uniquely, then the decision has been made on a maximum likelihood basis. In other words, the most probable, proper geometric guide and a specific part of it had been chosen each time, regarding the criteria introduced in Section 3. In the decisively greater number of brushstrokes, this maximum likelihood choice was, in practice, also unique. In the quite unusual cases, where there was not a predominant candidate for matching the contour of a brushstroke, then, one cannot be certain, if the corresponding contour segment had been drawn, let us say, by a blue hyperbola instead of a magenta one.
IV. The distance of the proper part of the geometric guide from the actually drawn contour line, is impressively small; we restate this minimum (mean) error is always less than 0.3 mm and, most frequently, less than 0.2 mm, while the corresponding maximum error is always smaller than 0.8 mm and, in practice, less than 0.65 mm. We firmly believe that these very low average and maximum errors, guaranteed the very stable line, which the prehistoric artist(s) desired his synthesis to have.

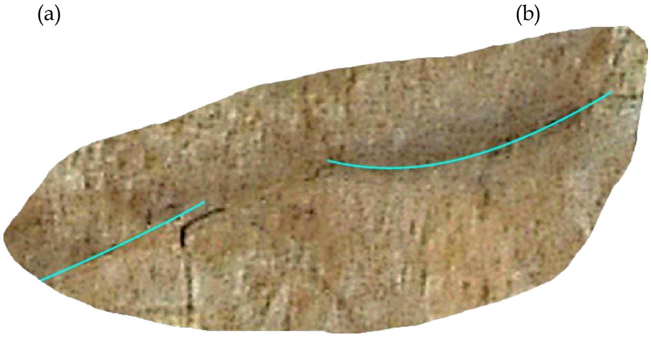
We stress in passing that this stability of the contour line appears in many, if not most, small brushstrokes, too, a fact indicating that guides, had, most probably, been used by the prehistoric artist(s) even for the drawing of particularly small contour parts. It is very logical to assume that the prehistoric artist(s) did so, because he had to generate very stable lines, on a particularly rough surface (like the one formed by plaster) in a very limited time period. On the contrary, certain connections and "junctions" had been sometimes drawn by free hand (see Fig. 26). In a future work, we shall try to give a rigorous mathematical distinction between contour parts that have been drawn via the use of a guide and those that they had been freely designed.

V. The particularly great degree of confidence that all contours of the figures of the objects appearing in the studied wall-paintings have been drawn via the six spotted geometric stencils, is strongly supported by the impressively large length of numerous corresponding contour parts. In fact, the authors have determined more than two hundred (200) stencils' parts of length greater than 8 cm and up to 32.8 cm, matching the actually drawn contours with an exceptionally low error. The aforementioned remarks decisively reduce the possibility that the spotted matching between the actually drawn contour parts and the optimally fitted geometric parts to them, is random or accidental. On the contrary, all previous observations and conclusions strongly support the hypothesis that the method of drawing of the studied Late Bronze Age wall-paintings, was the one presented here. After all, the essence of science lies in the fact that repetitions of the same results without exception, imply causality.

(a)



(b)



(c)

Figure 26. Some connections and "junctions" of three wall-paintings, namely: in (a) "Sea Daffodils"; in (b) "Rhyton Bearer"; in (c), "Prince of the Lilies". All these connections were drawn via free hand. The instability and fluctuation of the border line of the depicted "junctions" and "corrections" is more than evident. In addition, the undulation of the contour lines made by hand, is in a remarkable contrast with the impressive stability of the border lines generated via the use of stencils.

6.2. *Queries, Conjectures and Analysis Concerning the Origin(s) of the Proposed Method of Frescoes' Drawing*

Before anything else, we feel that we must emphatically comment on the "colossal" novelty associated with the eventual design method introduced here and, in particular, with the conception and construction of the specific geometric guides. In fact,

a) For persons that have studied Mathematics and Engineering quite extensively and do research in these scientific disciplines for years, it seems amazing, if not incredible and definitely extraordinary, the fact that an individual or a group of individuals had most probably conceived the shapes of the hyperbola and the linear spiral, in the Late Bronze Age. Even more impressive is the fact that this huge inspiration took place at least one thousand three hundred (1300) years before the so far, officially accepted conception of them by "giants" of Geometry and mathematical thought in the Classical Age, such as Archimedes (Αρχιμήδης ο Συρακόσιος), Menaechmus (Μέναιχμος), Apollonius (Απολλώνιος ο Περγεύς), Conon (Κόνων ο Σάμιος), Euclid (Ευκλείδης) and others.
b) Equally remarkable is the eventuality that this or another group of persons achieved in implementing the aforementioned geometric shapes with a precision and accuracy pretty close to the one a craftsman could accomplish today.
c) It is also impressive that someone conceived the idea of using a very limited number of geometric guides, as stencils, in order to draw all the border lines of a great variety of figures, motifs, art forms, subjects and syntheses.

At a next step, one may express numerous queries in connection with the specific geometric stencils and the method of drawing proposed here. Indeed,

i. Was a single person or a sequence of persons that had, most likely, conceived the shape of hyperbola (and perhaps the cone and the conic sections), as well as of the linear spirals? It is evident that any related hypothesis-conjecture can be proved very dangerous and risky; all the same, the authors feel committed to consider the evolution of the entire knowledge associated with Mathematics, Physics, Technology and all Exact Sciences, in general, throughout the historical period. In this historical route, novelty, as a rule, is an outcome of the inspiration of a single person or, at most, of a small group of persons; the greater the "amount" of the resulting novelty, the more probable is that this was a result of a greater inspiration of a single individual.

In numerous cases, the requirement for such a novelty is (strongly) associated with everyday needs, social and religious demands, wealth accumulation, etc. Hence, in the prehistoric era we deal with, it is plausible to assume that the novelty associated with the proposed method of frescoes' drawing, is intimately connected to religious, cult and mystical demands, and/or with social recognition and admiration and/or wealth accumulation.

ii. There is another very interesting question concerning this method of frescoes' drawing: why the inventor(s) of this method had chosen the hyperbola and the linear spiral as "the fundamental units of drawing", given that these curves do not exist in nature, nor they are encountered in everyday life? In fact, there are various spirals and numerous other geometric shapes, which emerge in everyday life and/or may be found in nature with an impressive precision, such as:
   a) The exponential spiral, formed with an impressive accuracy in the seashells.
   b) The unwinding spirals, namely the shape that is generated, when one unwraps a rope initially wrapped around a peg. Clearly, such a spiral may emerge in various events of everyday life; a simple example is the traces of the footsteps of a domestic animal tied on a rope in a peg.
   c) The straight-line segment, which evidently appears when one stretches a rope or a thread or a long hair, etc., between two fixed points.
   d) The circle, which, for example, is generated when one turns an object of a fixed length around a fixed point.
   e) The ellipse, which emerges if one moves a brush being continually in contact with a rope or a thread, the end points of which are kept fixed, and other shapes.

iii. One cannot exclude the possibility that the person(s) who had conceived the hyperbolae and the linear spirals did so, exactly because these shapes did not fall into everyday experience. In other words, it is possible that the inventor(s) of the method wanted to convey a sense and feelings of mysticism, cult, religion, "supernatural powers", artistry etc.

iv. Of course, due to the lack of related archaeological evidence, we do not have the right to adopt the opinion that the conception of these geometric figures took place in an Aegean Island. However, the surprisingly great and remarkable amount of novelty associated with such a conception, seems to drastically reduce the probability that this inspiration took place much earlier than the flourishing of the Minoan Late Bronze Age civilizations. Such an inspiration must be strongly associated with a very advanced for the era civilization, like the ones that dominated the Aegean Sea in the specific prehistoric era.

v. The method of construction of the presumed stencils and/or of the apparatuses, is also impressively novel for the era, especially if one takes into consideration the amazing precision with which the proposed geometric guides match the actual brushstrokes. Again, there are numerous associated questions, such as:
   a) Were the same persons, who conceived the idea of using these geometric stencils for the wall-paintings' drawing, with those who constructed them?
   b) Did the construction take place in the Aegean for the first time or not? In any case, the incredibly good, piecewise matching of practically all brushstrokes' contours indicates an impressive craftmanship and technological knowledge in the aforementioned Late Bronze Age civilizations. We must emphasize that this impressive craftmanship is fully compatible with the considerably high technological level of the Akrotiri, Thera, and Minoan Crete civilizations, as it has been already pointed out by prof. C. Doumas and his collaborators (e.g., see (Doumas, 1992; Sakellarakēs and Doumas, 1994)).

vi. Was there a kind of School (in Thera, in Crete or somewhere else in the Aegean Sea) that taught the construction of these geometric prototypes, as well as their great usefulness in drawing practically any figure?

vii. Why these stencils/guides had not evolved and/or changed in the subsequent one thousand (1000) years? It is very plausible to assume that the "permanent" adoption of these guides and of the corresponding method of drawing, had been an inextricable part of a religion, a predominant mysticism and worship, etc.

Nothing in the mathematical analysis presented here, detracts from the humane value of these paintings. We strongly believe that the frescoes' artists were motivated by the sense of elegance, of real beauty, of "the joy for life", etc. The results of the present work do not detract from the creative accomplishment of this remarkably consistent and stable style of painting, but it may explain it. Observers of Minoan art have been struck by how the human and natural figures are organic. In identifying the recurring use of hyperbolae and Archimedean spirals-curves that do not appear in nature, we might explain how figures in this style can seem "organic".

Our approach is limited to the determination of the method of painting on surviving fragments, ignoring the confounding, modern reconstructions, made according to aesthetic principles, but unaware of the rigorous geometric basis for these works of art. In our firm opinion, the present work should be the basis for any further reconstruction, and in fact it ought to invite re-thinking of existing reconstructions.

The present work also invites reconsideration of the history of mathematical, and particularly geometric thought in the Hellenic World and in general. For example, scholarly attention to the considered "Fathers of Greek Mathematics", Thales of Miletus and Pythagoras of Samos, who both lived in the 600s BCE, has focused on the extent to which they were influenced by the Mesopotamians or Egyptians (Robson 1999). However, the sophistication and rigor evidence in the construction of these templates that we describe here, in the Late Bronze Age Minoan period, a thousand years before Thales and Pythagoras, suggest that in addition to any Near Eastern or Egyptian influence on their thought, a well-established body of geometric understanding had permeated their culture for centuries.

Finally, the authors strongly believe that the following conclusive result is unambiguous: "The collective subconscious" of the rigorous Geometry developed in the Classical and Hellenistic Ages, as well as of contemporary Mathematics and of numerous Exact Sciences definitely existed in the Late Bronze Age Civilizations that had flourished in the Aegean Sea.

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SUPPLEMENTARY

S.M.1. Introduction

Observed Repetitions in Various Wall-Paintings

Thus, for example:

Crowley (Crowley, 1997) has, in an astute manner, observed noteworthy repetitions among spiral themes appearing in Minoan frescoes. She, furthermore, wonders if the Minoans knew basic principles and notions of Geometry and if so, up to what extent. It is worthwhile noticing that the authors of the present work had set the same questions in the very same period; however, the development of a rigorous and consistent Mathematical methodology, together with a corresponding Information System that would answer these queries, is/was a tedious, time consuming and by no means trivial procedure. For this reason, the publication of the associated results required many years, up to twenty-five (25).

Moreover, we immediately below cite a set of important remarks made in (Bietak Manfred et al., 2007): "We found out that the measures and outlines of the body parts of all bulls were nearly identical. We discovered, for example, that we could superimpose two bulls and get identical outlines of equivalent body parts. Although the postures of the animals varied, the proportion and the general form were constant within the same painting. We became convinced that the Minoan artists themselves used some kind of template (perhaps a schematic drawing or a cut-out of parchment) to ensure standard forms and exact proportions. … Yet, since the proportions of the animals are identical, the concept of a template may be acceptable. It must be stressed that we understand the template to have been used as a guide for proportions and that the execution of details was left to the individual skill of each artist; in fact, there is no doubt that there was more than one artist at work for each wall-painting."

S.M.2. Establishing the Method of Drawing of a Set of Four (4) Frescoes, Excavated at Akrotiri Thera Belonging to the Late Bronze Age

S.M.2.1 Concerning the Fresco "Sea Daffodils" or "Lilies"

Immediately below, we give a brief description of this wall-painting, fully respecting the associated content of (Doumas, 1992): "the lower zone of this wall-painting comprises a broad reddish-yellow surface with an undulating upper limit, presumably an attempt at rendering an uneven ground. The upper zone consists of a system of narrow black, red and blue bands alternating with white. Between these two zones the main theme of the middle zone is developed: a representation of blossoming plants growing out of the uneven ground." An image of this synthesis is shown in Fig. S.M-1; the image has been scanned from the reference book (Doumas, 1992).

In connection with the question about the plants' species appearing in this fresco, two major, different opinions have been expressed so far: S. Marinatos believed that the plants are the "Pancratium lily" (Pancratium Maritimum). On the contrary, later on, P. Warren suggested that the plants were papyri, a familiar subject in Minoan art.

However, after the study referred to in (Baumann et al., 1993), most probably the plant depicted in the wall-painting is indeed the "Pancratium lily", a "sea Daffodil". A sub-group of the authors has a rather good familiarity with/knowledge of the flora of Cycladic islands and, consequently, are in a position to agree with and confirm the opinion of S. Marinatos and Prof. C. Doumas. In any case, the plants depicted in this wall-painting (see Fig. S.M-1) certainly are a magnification of the natural ones.

Figure S.M-1. An image of the synthesis "Sea Daffodils", scanned from (Doumas, 1992).

S.M.2.2 Study of the Method of Drawing of the Figure "The Griffin" Belonging to the Synthesis "The Crocus Gathering"

A short presentation of this fresco, entirely based on (Doumas, 1992) follows (see also Fig. S.M-2):

Figure S.M-2. An image of the mythical creature "Griffin", scanned from (Doumas, 1992).

The upper and the lower zone comprised horizontal bands and the main theme, the "gathering of crocus" ("Κροκοσυλλέκτριες"), was developed in the middle zone. In fact, in a rocky, mountainous landscape scattered with clusters of crocuses, four (4) female figures are engaged/ absorbed in the collection of this valuable commodity. At the center of the representation is a majestic female figure, most probably a goddess, seated on a stepped structure. The figure is flanked in the left by a blue monkey and in the right by a griffin (γρὐπας). Although the lower part of the griffin is not well-preserved, its pose is clear. It, too, is presented as if climbing up to the seated figure, its front legs placed on the stepped structure. The animal seems to be tied with a rope which is partially covered by the unfolded wing and it is terminated in the upper right corner of the representation, presumably attached to something. An image of the entire synthesis can be found in pp. 158-159 of (Doumas, 1992).

Figure S.M-3. The detected optimal parts of the geometric guides, without the underlying "griffin" image.

Figure S.M-4. The contour of the figure depicted in image S.M-2 with all corresponding unique guides' parts on it.

Table S.M-1. The standard numerical characteristics of the determined geometric guides' parts, for each type of guide separately a) concerning all guides' parts, b) regarding only the unique ones.

Stencil Type	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	11	0	2.80	-	13.19	-	0.10	-	0.32	-
c	201	15	0.64	2.28	18.38	16.72	0.10	0.16	0.27	0.46
g	22	4	1.90	6.80	17.39	17.39	0.12	0.21	0.35	0.59
m	1	0	2.56	-	2.56	-	0.04	-	0.11	-
ls0	51	-	0.62	-	2.12	-	0.11	-	0.33	-
ls2	21	0	0.23	-	9.34	-	0.07	-	0.18	-

S.M.2.3. How the middle figure of the synthesis "Adorants" in the lustral basin of Xeste 3 had been drawn

Again, following the book by Prof. C. Doumas (Doumas, 1992), below we describe the middle figure of the wall-painting "The Adorants". In fact, this room is arranged as a "Lustral Basin", an installation known from the so called "Minoan palaces". In the middle zone, which on the north wall extended westwards beyond the Lustral Basin, the artist(s) developed the main theme of the aforementioned wall-painting.

The entire composition, as restored, depicts the following: on the east wall was a structure surmounted by a pair of sacral horns from the tips of which drip red drops, probably blood. This part of the synthesis is, nowadays, in a fragmentary condition. On the north wall, three female figures process towards the "altar"; for this reason, these figures today bear the name "Adorants" (Λατρεὺτριες). Their rich Minoan garments, their elaborate coiffures and ornate jewelry of precious metals and rare gems, not only manifests the festive character of the scene, but also reveal the status of these women in the Theran society (Doumas, 1992).

The middle female figure, an image of which is shown in Fig. S.M-5 of the present work, is depicted entirely in profile, sitting on a small knoll, and slightly bent over.

Figure S.M-5. A female figure belonging to the synthesis "Adorants", which was decorating the walls of the "Lustral Basin" in Xeste 3 (photo scanned from (Doumas, 1992)).

Figure S.M-6. The parts of the stencils that cover the entire contour of the very same female "Adorant", shown in Figure 18 of the main text, standing alone.

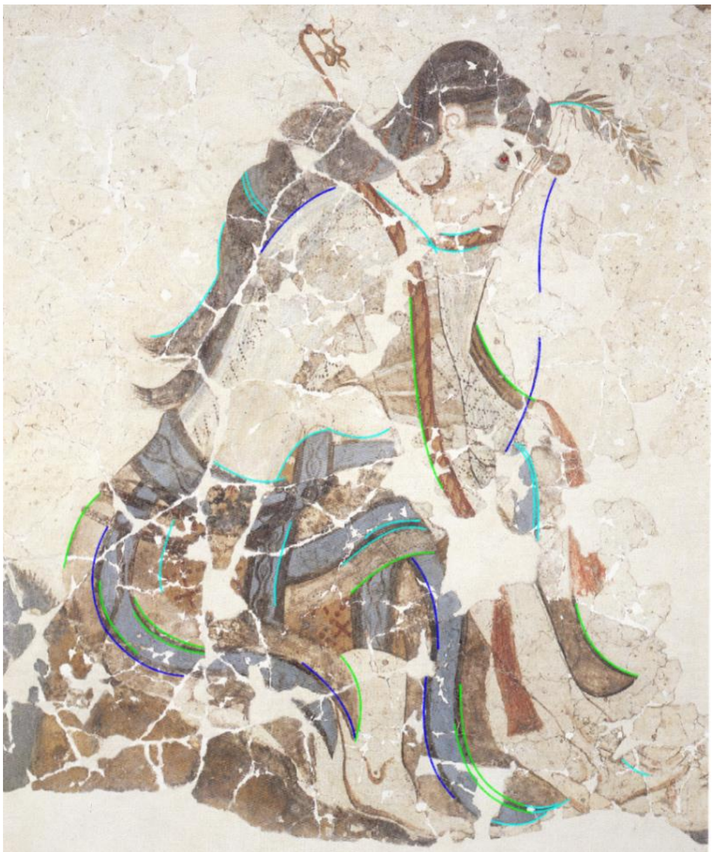


Figure S.M-7. The subset of the geometric guides' parts shown in Figure 18 of the main text, which are unique.

Table S.M-2. The already adopted numerical characteristics of the determined geometric stencils' parts in connection with the wall-painting shown in Figures 18 (main text) and in S.M-7.

Type of Stencil	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	25	8	5.13	10.84	15.11	14.73	0.16	0.22	0.49	0.6
c	85	21	1.64	3.68	11.75	11.61	0.13	0.17	0.4	0.48
g	33	11	3.26	9.69	22.37	22.37	0.15	0.2	0.41	0.53
m	0	0	-	-	-	-	-	-	-	-
ls0	13	0	0.83	-	2.45	-	0.1	-	0.29	-
ls2	186	1	0.37	1.19	8.92	1.19	0.09	0.22	0.26	0.5

S.M.2.4. The Corresponding Analyses Concerning the Wall-Painting Named "Fisherman"

Following Doumas (Doumas, 1992), the specific wall-painting depicts a young, nude male figure, his head and lower limbs shown in profile, the chest "en face" and the abdomen in three-quarter pose (Fig. S.M-8). Apart from two black tresses, one at the front and one behind, the entire head is painted blue, probably due to a certain convention. By showing the arms open to the sides, the artist, most probably, overcame the problem of confusing the left and right hand. Indeed, he rendered both thumbs pressed in, trying to avoid perspective difficulties and in addition, probably wanting to depict the figure "en face", with both arms extended in front. The young fisherman holds a bunch of fish in each hand, seven in the right and five in the left. Three colors have been used for each fish: black for the outline and fins, yellow for the belly and blue for the back.

We note that in Figures S.M-9 (a), (b) and S.M-10 three (3) sub-images with greater resolution of this painting are presented.

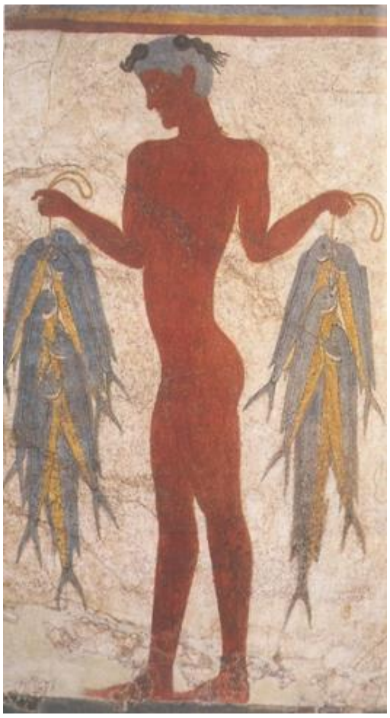


Figure S.M-8. The image of the "Fisherman" (photo scanned from (Doumas, 1992)).

(a) (b)

Figure S.M-9. In (a) the left ensemble of fishes is depicted, while in (b) the right set of fishes is shown.

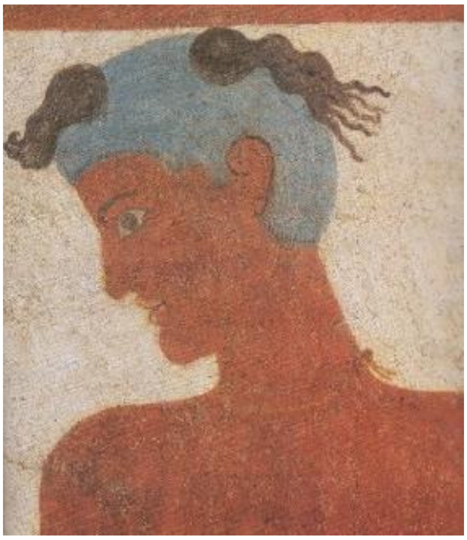


Figure S.M-10. The head of this fisherman is presented with a greater image resolution.

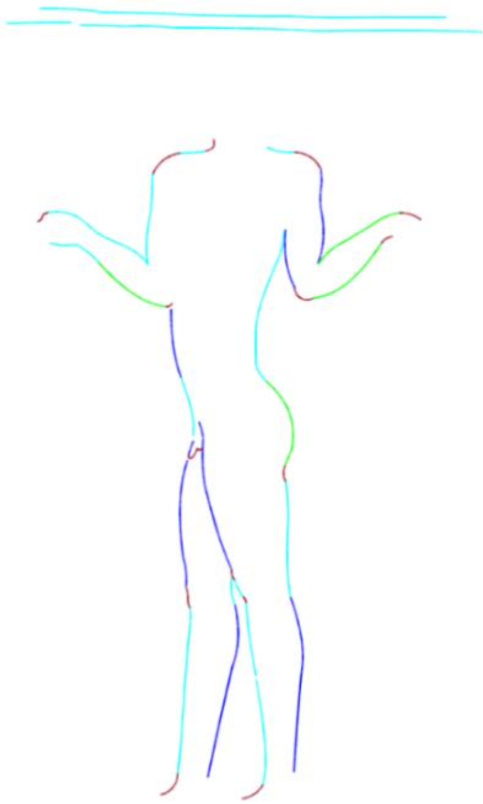


Figure S.M-11. The parts of the geometric guides of Figure 19 of the main text, standing alone.

(a) (b)

Figure S.M-12. The parts of the geometric guides of Figures 20 (b) and (c) of the main text, standing alone.

Figure S.M-13. The parts of the geometric guides of Figure 20 (a) standing alone.

(a) (b)

Figure S.M-14. The contour of the figure depicted in images 19 and 20 (c) of the main text with all corresponding unique stencils parts on it, in (a) and (b) respectively.

Table S.M-3. The classical, by now, numerical characteristics of the determined geometric guides' parts, in connection with all the aforementioned Figures.

Type of Stencil	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	15	0	2.06	-	15.54	-	0.13	-	0.36	-
c	132	7	0.62	6.34	22.74	21.26	0.1	0.23	0.28	0.62
g	25	2	1.94	10.43	15.5	12.30	0.13	0.17	0.35	0.50
m	0	0	-	-	-	-	-	-	-	-
ls0	24	0	0.33	-	2.94	-	0.1	-	0.47	-
ls2	329	0	0.16	-	6.41	-	0.08	-	0.2	-

S.M.3. Demonstration that Many Minoan, Cretan Frescoes Had Been Drawn by the Same Method and Guides as the Previous Ones of Akrotiri

S.M.3.1. Study of the wall-painting "The Cup-bearer" or "The Rhyton-Bearer"

In the following, a brief description of this fresco is given, based on the books (Sakellarakēs and Ntumas, 1994) and (Dimopoulou - Rethemiotaki, Nota, 2005):

The body of this youthful male figure is depicted in the conventional red color (see image S.M-15). The gaze of this young person is fixed, his hair is richly dressed, while he wears an elaborate loincloth and jewelry and holds a tall, conical rhyton in both hands. His pose is statuesque, all the same, the young man conveys the importance of his mission, in a convincing and effective manner. The figure is crowned by a "rock", most probably, conventionally indicating the landscape.

Figure S.M-15. The wall-painting the "Rython Bearer" (photo scanned from book (Dimopoulou - Rethemiotaki, Nota, 2005)).

(a) (b)

(c)

Figure S.M-16. (a) An image of the upper part of the "Rython Bearer", having a substantially greater resolution; (b) An image with a considerably greater analysis than the one of S.M-15, depicting the middle part of the "Rython Bearer"; (c) An image with a considerably greater analysis than the one of S.M-15, depicting the lower part of the "Rython Bearer".

S.M.3.2. Regarding the wall-painting(s) "the Prince of Lilies"

We would like to emphasize that, in the assemblage of Fig. S.M-17, the figure walks left, against a red background, and perhaps holds a griffin or sphinx. He wears a loincloth with a broad belt, a necklace and an elaborate diadem with lilies and peacock feathers.

However, Shaw in (Shaw, 2004), claims that the direction to which the figure was originally supposed to move is unknown/ambiguous; indeed, it seems that Evans and his restorers took quite many unfounded initiatives in the restoration of the synthesis.

Furthermore, Niemeier (Niemeier, 1986) argued that the separate fragments which Evans and his restorers attributed to a single synthesis, most probably belong to more than one figures. This is supported by the fact that the depicted crown was of a type normally worn by female figures. Moreover, the color of the skin strongly suggests that the various excavated fragments initially belonged to figures of different sex. The analysis presented in (Niemeier, 1986; Shaw, 2000) received a broad acceptance. For this reason, the authors of the manuscript in hand show three different well-preserved "islands" of fragments in Figures S.M-18 (a), (b) and (c), which they have called by the authors "top island", "middle island" and "lower island". We emphasize that in these "islands", we have not included the interventions made by Evans and his collaborators/restorators.

In the top island (S.M-18 (a)), there is an elaborate head-dress decorated with lilies and peacock feathers. In addition, despite the poor state of preservation of the fragments, the artist's effort to render the muscles of the depicted figure and the details of the garment in the middle and lower islands (S.M-18 (b) and S.M-18 (c), respectively) is evident.

Finally, concerning the period during which these fragments (islands) were drawn, Hood in (Hood, 2005) states that all of them belong to Late Minoan IA or Late Minoan IB, in agreement with Evans and Cameron (Cameron, 1976; Evans, n.d.) agreed with these dates.

Figure S.M-17. An image of the fresco "Prince of the Lilies" (photo scanned from book (Dimopoulou - Rethemiotaki, Nota, 2005)). We emphasize that this fresco was a result of a number of unfounded assumptions of Evans and his collaborators. For this reason, we have divided it into three fragments shown in Figures S.M-18 (a), (b) and (c).

(a)

(b)

(c)

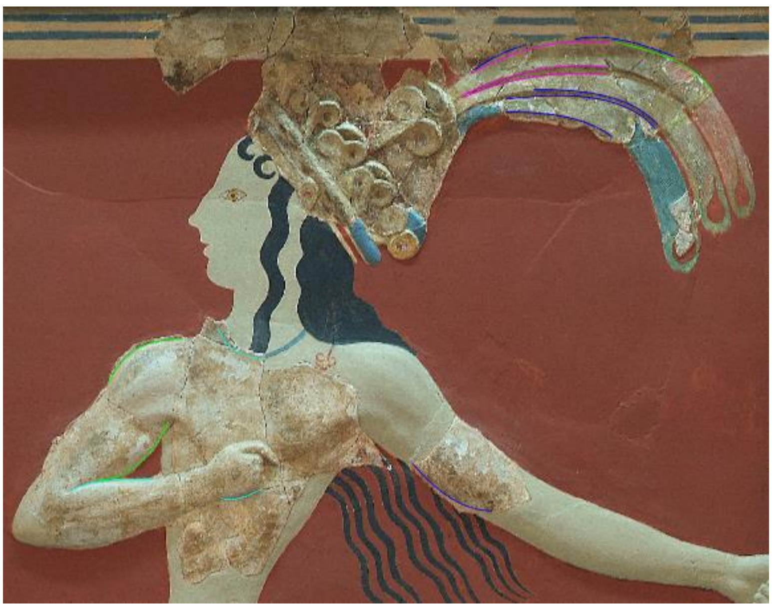
Figure S.M-18. The three fragments to which the "Prince of Lilies" has been separated; (a) The upper fragment of the fresco is depicted, which we call "The top island"; (b) The middle part of the "prince" is shown and we call it "The middle island"; (c) "The lower island" of the synthesis.

(a)

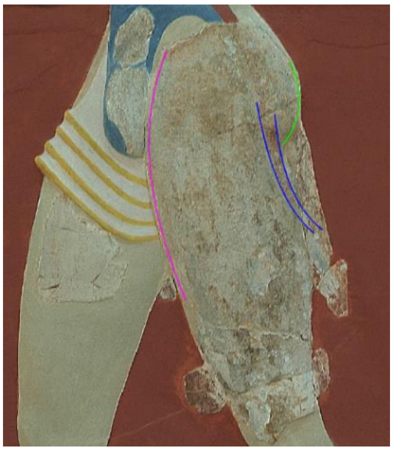
(b)

(c)

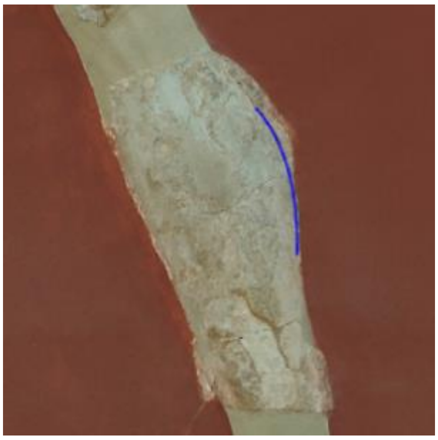
Figure S.M-19. (a) The parts of the geometric guides of Figure 24 (a) of the main text standing alone; (b) The parts of the geometric guides of Figure 24 (b) standing alone; (c) The parts of the geometric guides of Figure 24 (c), covering the lower island, standing alone.



(a)



(b)



(c)

Figure S.M-20. (a) The subset of the geometric guides, which fully cover the top island (Figure 24 (a)) that are unique; (b) The unique guide parts of the middle island (Figure 24 (b)); (c) unique guide parts (Figure 24 (c)) of the lower island.

Table S.M-4. The usual numerical characteristics associated with Figures 24 (a) and S.M-20 (a).

Type of Stencil	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	11	2	8.17	8.17	22.3	22.3	0.23	0.25	0.59	0.69
c	9	8	1.3	6.25	14.2	12.01	0.16	0.19	0.44	0.53
g	13	2	2.65	12.95	15.72	15.72	0.19	0.26	0.51	0.65
m	3	0	20.39	20.39	21.25	21.25	0.24	0.24	0.71	0.71
ls2	51	0	0.8	1.78	10.83	1.78	0.17	0.08	0.5	0.35
ls0	11	2	1.53	-	3.52	-	0.16	-	0.41	-

Table S.M-5. The adopted numerical characteristics in connection with the middle island (Figures 24 (b) and S.M-20 (b)).

Type of Stencil	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	5	2	10.6	16.49	19.31	18.66	0.19	0.23	0.54	0.59
c	0	0	-	-	-	-	-	-	-	-
g	1	1	11.51	11.51	11.51	11.51	0.29	0.29	0.64	0.64
m	2	1	12.42	32.83	32.83	32.83	0.22	0.29	0.62	0.77
ls2	0	0	-	-	-	-	-	-	-	-
ls0	0	0	-	-	-	-	-	-	-	-

Table S.M-6. The already standard numerical characteristics in connection with "the lower island" (Figures 24 (c)) and S.M-20 (c)).

Type of Stencil	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	4	1	5.03	12.76	13.95	12.76	0.07	0.11	0.18	0.24
c	0	0	-	-	-	-	-	-	-	-
g	0	0	-	-	-	-	-	-	-	-
m	0	0	-	-	-	-	-	-	-	-
ls2	0	0	-	-	-	-	-	-	-	-
ls0	0	0	-	-	-	-	-	-	-	-

S.M.3.3. Establishing the Method of Drawing of the Minoan Fresco "The Blue Bird"

Below, we shall give a short presentation of the specific assemblage, following (Cameron, 1976; Hood, 2005; Sakellarakēs and Ntumas, 1994) (see also Fig. S.M-21 below):

Among a large ensemble of blossoms, such as bright yellow white roses, pale blue sweet peas, lilies, etc., a blue bird sits on a rock against a white ground. The bird's wings are painted blue and black, while the rest of the body was probably yellow. Though the figure is, unfortunately, not completely preserved, the position of its wings and its short legs suggest that the bird is relaxing. According to (Sakellarakēs and Ntumas, 1994), "this fragment, distinguished by the free brushstrokes, the revelry of nature and its restfulness to the eye, must have belonged to a magnificent assemblage which was destroyed". Concerning the bird species, Evans (Evans, n.d.) guessed it was a swallow, but Mackenzie believed that the representation belonged to a bird of passage of another species, still found in Crete (Hood, 2005).

Figure S.M-21. The wall-painting "The blue bird"(photo scanned from (Sakellarakēs and Ntumas, 1994)).

Figure S.M-22. The parts of the geometric guides of Figure 25 of the main text, standing alone. It is obvious that these geometric sub-curves generate a "skeleton" of the contours of S.M-21, in an excellent manner.

Figure S.M-23. The parts of the geometric guides, which fully cover the Figure 25 of the main text that are unique.

Table S.M-7. The adopted numerical characteristics of fitting, in connection with the fresco "The blue bird".

Type of Stencil	Number of Occurrences		Minimum Length (cm)		Maximum Length (cm)		average of minimum errors (mm)		average of maximum errors (mm)	
	Total	Unique	Total	Unique	Total	Unique	Total	Unique	Total	Unique
b	0	0	-	-	-	-	-	-	-	-
c	9	8	1.54	3.65	7.81	7.81	0.18	0.19	0.52	0.57
g	7	2	2.13	7.22	8.45	8.45	0.14	0.18	0.37	0.5
m	0	0	-	-	-	-	-	-	-	-
ls2	40	0	1.3	-	9.9	-	0.14	-	0.43	-
ls0	0	0	-	-	-	-	-	-	-	-